問題 5.1 $t = \log(1+u)$ とおくと $e^t = 1+u$

$$\lim_{u\to 0}\frac{\log(1+u)}{u} = \lim_{t\to 0}\frac{t}{e^t-1} = \lim_{t\to 0}\frac{1}{\dfrac{e^t-1}{t}} = \frac{1}{1} = 1$$

問題 5.2

(1)

$$(\log(-x))' = \frac{-1}{-x} = \frac{1}{x}$$

(2)

$$\left(\frac{e^x+e^{-x}}{2}\right)' = \frac{e^x-e^{-x}}{2}$$

(3)

$$(x\log x)' = 1\log x + x\cdot\frac{1}{x} = \log x + 1$$

(4)

$$\left(\frac{x}{e^x}\right)' = \frac{1e^x - xe^x}{(e^x)^2} = \frac{e^x(1-x)}{(e^x)^2} = \frac{1-x}{e^x}$$

問題 5.3

(1)

$$(\cos x)' = \left(\sin\left(\frac{\pi}{2}-x\right)\right)' = -1\cos\left(\frac{\pi}{2}-x\right) = -\sin x$$

(2)

$$\begin{aligned}(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \begin{cases}\dfrac{1}{\cos^2 x}\\ 1+\dfrac{\sin^2 x}{\cos^2 x} = 1+\tan^2 x\end{cases}\end{aligned}$$

問題 5.4

(1)

$$(\log|\cos x|)' = \frac{-\sin x}{\cos x} = -\tan x$$

(2)

$$\left(e^{\sin(2x)}\right)' = 2\cos(2x)e^{\sin(2x)}$$

(3)

$$(e^{-x}(\cos x+\sin x))' = -e^{-x}(\cos x+\sin x) + e^{-x}(-\sin x+\cos x) = -2e^{-x}\sin x$$

問題 5.5

(1) $y = \cos^{-1} x \iff x = \cos y$ かつ $0 \leqq y \leqq \pi$

$$\therefore \quad \left(\cos^{-1} x\right)' = \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{(\cos y)'} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}}$$

(2) $y = \tan^{-1} x \iff x = \tan y$ かつ $\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

$$\therefore \quad \left(\tan^{-1} x\right)' = \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{(\tan y)'} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

問題 5.6

(1)

$$\left(\tan^{-1}\left(\frac{x}{2}\right)\right)' = \frac{1}{2}\frac{1}{1+\left(\dfrac{x}{2}\right)^2} = \frac{2}{4+x^2}$$

(2)

$$\left(\sin^{-1}\left(\frac{x}{\sqrt{3}}\right)\right)' = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{1-\left(\dfrac{x}{\sqrt{3}}\right)^2}} = \frac{1}{\sqrt{3-x^2}}$$

(3) $t = \sqrt{x}$ とおくと $y = \cos^{-1} t$

$$\left(\cos^{-1}\sqrt{x}\right)' = \frac{dy}{dt}\frac{dt}{dx} = -\frac{1}{\sqrt{1-t^2}}\frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{(1-x)x}}$$

問題 5.7

(1) $y = \log\left(x+\sqrt{x^2-1}\right) \iff x = \dfrac{e^y+e^{-y}}{2}$

$$\left(\log\left(x+\sqrt{x^2-1}\right)\right)' = \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{\dfrac{e^y-e^{-y}}{2}} = \frac{1}{\sqrt{x^2-1}}$$

(2) $y = \log\left(x+\sqrt{x^2+1}\right) \iff x = \dfrac{e^y-e^{-y}}{2}$

$$\left(\log\left(x+\sqrt{x^2+1}\right)\right)' = \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{\dfrac{e^y+e^{-y}}{2}} = \frac{1}{\sqrt{x^2+1}}$$

一般に,

公式 5.1

$$\left(\log\left(x+\sqrt{x^2+A}\right)\right)' = \frac{1}{\sqrt{x^2+A}}$$