6 $y' = f(ax+by+c)$ の形

$y' = f(ax+by+c)$ の形に変形できる微分方程式は，$u = ax+by+c$ とおくと，u に関する変数分離形の微分方程式が得られる。その解から元の方程式の解を求めればよい。

例 $y' = 4x^2 - 4xy + y^2 + 4x - 2y - 1$

$$y' = (2x-y)^2 + 2(2x-y) - 1$$

$u = 2x - y$ とおくと

$$\begin{aligned}
u' &= 2 - y' \\
&= 2 - (u^2 + 2u - 1) \\
&= -(u^2 + 2u - 3) \\
\therefore\ 1 &= \frac{-1}{u^2+2u-3}u' \\
&= \frac{-1}{(u+3)(u-1)}u' \\
&= \left(\frac{A}{u+3} + \frac{B}{u-1}\right)u' \\
&= \frac{1}{4}\left(\frac{1}{u+3} - \frac{1}{u-1}\right)u'
\end{aligned}$$

$$\begin{aligned}
\int 1\,dx &= \int \frac{1}{4}\left(\frac{1}{u+3} - \frac{1}{u-1}\right)du \\
x + C_1 &= \frac{1}{4}\left(\log|u+3| - \log|u-1|\right) \\
4x + C_2 &= \log\left|\frac{u+3}{u-1}\right| \\
\pm e^{4x+C_2} &= \frac{u+3}{u-1} \\
Ce^{4x} &= 1 + \frac{4}{u-1} \\
u - 1 &= \frac{4}{Ce^{4x}-1} \\
2x - y - 1 &= \frac{4}{Ce^{4x}-1} \\
y &= 2x - 1 - \frac{4}{Ce^{4x}-1}
\end{aligned}$$

問題

(1) $y' = (x+y)^3 - x - y - 1$

(2) $y' = \sin^2(x-y)$

(3) $(2x-y)y' = 1$

(4) $y' = e^{x-2y+3}$

解答

(1) $y' = (x+y)^3 - x - y - 1$

$$\begin{aligned} y' &= (x+y)^3 - (x+y) - 1 \end{aligned}$$

$u = x + y$ とおく

$$\begin{aligned} u' &= 1 + y' \\ &= 1 + u^3 - u - 1 \\ &= u^3 - u \\ &= u(u-1)(u+1) \\ \therefore\ 1 &= \frac{1}{u(u-1)(u+1)}u' \\ &= \left(\frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1}\right)u' \\ &= \left(-\frac{1}{u} + \frac{\frac{1}{2}}{u-1} + \frac{\frac{1}{2}}{u+1}\right)u' \end{aligned}$$

$$\begin{aligned} \int 2\,dx &= \int \left(-\frac{2}{u} + \frac{1}{u-1} + \frac{1}{u+1}\right)du \\ 2x + C_1 &= -2\log|u| + \log|u-1| + \log|u+1| \\ &= \log\left|\frac{(u-1)(u+1)}{u^2}\right| \\ &= \log\left|\frac{u^2-1}{u^2}\right| \\ \pm e^{2x+C_1} &= \frac{u^2-1}{u^2} \\ Ce^{2x} &= 1 - \frac{1}{u^2} \\ u^2 &= \frac{1}{1-Ce^{2x}} \\ (x+y)^2 &= \frac{1}{1-Ce^{2x}} \end{aligned}$$

(2) $y' = \sin^2(x-y)$

$u = x - y$ とおく

$$\begin{aligned} u' &= 1 - y' \\ &= 1 - \sin^2 u \\ &= \cos^2 u \\ \therefore\ 1 &= \frac{1}{\cos^2 u}du \end{aligned}$$

$$\begin{aligned} \int 1\,dx &= \int \frac{1}{\cos^2 u}\,du \\ x + C_1 &= \tan u \\ x + C &= \tan(x-y) \end{aligned}$$

(3) $(2x-y)y'=1$

$$
\begin{aligned}
y' &= \frac{1}{2x-y}
\end{aligned}
$$

$u=2x-y$ とおく

$$
\begin{aligned}
u' &= 2-y' \\
&= 2-\frac{1}{u} \\
&= \frac{2u-1}{u} \\
\therefore\ 1 &= \frac{u}{2u-1}u'
\end{aligned}
$$

$$
\begin{aligned}
\int 1\,dx &= \int \frac{u}{2u-1}\,du \\
x+C_1 &= \int \left(\frac{1}{2}+\frac{\frac{1}{2}}{2u-1}\right)du \\
&= \frac{1}{2}u+\frac{1}{4}\log|2u-1| \\
4x+4C_1 &= 2u+\log|2u-1| \\
4x+C_2 &= 2(2x-y)+\log|4x-2y-1| \\
2y+C_2 &= \log|4x-2y-1| \\
\pm e^{2y+C_2} &= 4x-2y-1 \\
Ce^{2y} &= 4x-2y-1 \\
4x &= Ce^{2y}+2y+1
\end{aligned}
$$

(4) $y'=e^{x-2y+3}$

$u=x-2y+3$ とおく

$$
\begin{aligned}
u' &= 1-2y' \\
&= 1-2e^{u} \\
\therefore\ 1 &= \frac{1}{1-2e^{u}}u'
\end{aligned}
$$

$$
\begin{aligned}
\int 1\,dx &= \int \frac{1}{1-2e^{u}}\,du \\
x+C_1 &= \int \frac{e^{-u}}{e^{-u}-2}\,du \\
&= -\log|e^{-u}-2| \\
\pm e^{-x-C_1} &= e^{-u}-2 \\
Ce^{-x} &= e^{-x+2y-3}-2 \\
-x+2y-3 &= \log(Ce^{-x}+2) \\
2y &= \log(Ce^{-x}+2)+x+3
\end{aligned}
$$