7 同次形

$y' = f\left(\dfrac{y}{x}\right)$ の形に変形できる微分方程式を**同次形**という。
同次形の微分方程式は，$u = \dfrac{y}{x}$ とおくと，下記のように u に関して変数分離形になる。

$$
\begin{aligned}
u' &= \frac{y'x - y}{x^2} = \frac{y' - \dfrac{y}{x}}{x} = \frac{f(u) - u}{x} \\
\therefore\ \frac{1}{x} &= \frac{1}{f(u) - u} u'
\end{aligned}
$$

例 1. $(2x^2 + xy)y' = xy - y^2$

両辺を x^2 で割ると

$$
\left(2 + \frac{y}{x}\right) y' = \frac{y}{x} - \left(\frac{y}{x}\right)^2
$$

$$
\therefore\quad y' = \frac{\dfrac{y}{x} - \left(\dfrac{y}{x}\right)^2}{2 + \dfrac{y}{x}}
$$

$u = \dfrac{y}{x}$ とおくと

$$
\begin{aligned}
u' &= \frac{y'x - y}{x^2} = \frac{y' - u}{x} \\
\frac{1}{x} &= \frac{1}{\dfrac{u - u^2}{2 + u} - u} u' \\
&= \frac{2 + u}{u - u^2 - 2u - u^2} u' \\
&= \frac{2 + u}{-u - 2u^2} u' \\
&= \frac{-2 - u}{u(1 + 2u)} u' \\
&= \left(\frac{-2}{u} + \frac{3}{1 + 2u}\right) u'
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{1}{x}\, dx &= \int \left(\frac{-2}{u} + \frac{3}{1 + 2u}\right) du \\
\log|x| + C_1 &= -2\log|u| + \frac{3}{2}\log|1 + 2u| \\
2C_1 &= -4\log|u| + 3\log|1 + 2u| - 2\log|x| \\
&= \log\left|\frac{(1 + 2u)^3}{u^4 x^2}\right| \\
\therefore\ C &= \frac{(1 + 2u)^3}{u^4 x^2} \\
&= \frac{\left(1 + 2\dfrac{y}{x}\right)^3}{\left(\dfrac{y}{x}\right)^4 x^2} \\
&= \frac{\dfrac{(x + 2y)^3}{x^3}}{\dfrac{y^4}{x^2}} \\
&= \frac{(x + 2y)^3}{xy^4} \\
Cxy^4 &= (x + 2y)^3
\end{aligned}
$$

問題 次の微分方程式を解きなさい。

(1) $xy' = x + 3y$

(2) $(x + y)y' = 4y - 2x$

(3) $(2x^2 + y^2)y' = 2xy$

(4) $xy' = y(\log y - \log x)$

例 2. 4 機のミサイルの問題。

正方形の 4 つの頂点 A(1, 0), B(0, 1), C(−1, 0), D(0, −1) にミサイル M_A, M_B, M_C, M_D がある。ミサイル M_A はミサイル M_B を標的に設定されていて，常に M_B に向かって進んでいく。同様に，M_B は M_C に向かって，M_C は M_D に向かって，M_D は M_A に向かって進む。4 つのミサイルが同時に発射され等しい速度で追撃を始めると，いつか中心 O で激突して共倒れになる。

ミサイル M_A の軌跡を求めよ。

第 3 回に求めたように，この軌跡は次の微分方程式を満たす。

$$\begin{aligned} y' &= \frac{y-x}{y+x} \\ \therefore\ y' &= \frac{\frac{y}{x}-1}{\frac{y}{x}+1} \end{aligned}$$

$u = \dfrac{y}{x}$ とおくと

$$\begin{aligned} u' &= \frac{y'x-y}{x^2} = \frac{y'-u}{x} \\ \frac{1}{x} &= \frac{1}{\frac{u-1}{u+1}-u}u' \\ &= \frac{u+1}{u-1-u^2-u}u' \\ -\frac{1}{x} &= \frac{u+1}{1+u^2}u' \\ &= \left(\frac{u}{1+u^2}+\frac{1}{1+u^2}\right)u' \end{aligned}$$

$$\begin{aligned} -\int \frac{1}{x}\,dx &= \int \left(\frac{u}{1+u^2}+\frac{1}{1+u^2}\right)du \\ -\log|x|+C_1 &= \frac{1}{2}\log\left|1+u^2\right| + \tan^{-1}u \\ \therefore\ C_1 &= \frac{1}{2}\left(\log\left|1+u^2\right| + 2\log|x|\right) + \tan^{-1}u \\ &= \log\sqrt{(1+u^2)x^2} + \tan^{-1}u \\ &= \log\sqrt{\left(1+\left(\frac{y}{x}\right)^2\right)x^2} + \tan^{-1}\frac{y}{x} \\ &= \log\sqrt{x^2+y^2} + \tan^{-1}\frac{y}{x} \end{aligned}$$

極座標 $r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\dfrac{y}{x}$ で表すと

$$\begin{aligned} \log r &= -\theta + C_1 \\ \therefore\ r &= e^{-\theta+C_1} = Ce^{-\theta} \end{aligned}$$

これは**等角螺旋**(らせん)という曲線である。カタツムリや巻貝も等角螺旋である。

解答

(1) $xy' = x + 3y$

$$y' = 1 + 3\frac{y}{x}$$

$u = \dfrac{y}{x}$ とおく

$$\begin{aligned}
u' &= \frac{y'x - y}{x^2} = \frac{y' - u}{x} \\
\frac{1}{x} &= \frac{1}{y' - u}u' \\
&= \frac{1}{1 + 3u - u}u' \\
&= \frac{1}{1 + 2u}u' \\
\int \frac{1}{x}\,dx &= \int \frac{1}{1 + 2u}\,du
\end{aligned}$$

$$\begin{aligned}
\log|x| + C_1 &= \frac{1}{2}\log|1 + 2u| \\
2C_1 &= \log|1 + 2u| - 2\log|x| \\
&= \log\left|\frac{1 + 2u}{x^2}\right| \\
C_2 &= \frac{1 + 2u}{x^2} \\
&= \frac{x + 2y}{x^3} \\
2y &= C_2x^3 - x \\
y &= Cx^3 - \frac{1}{2}x
\end{aligned}$$

(2) $(x + y)y' = 4y - 2x$

$$\begin{aligned}
y' &= \frac{4y - 2x}{x + y} \\
&= \frac{4\frac{y}{x} - 2}{1 + \frac{y}{x}}
\end{aligned}$$

$u = \dfrac{y}{x}$ とおくと

$$\begin{aligned}
u' &= \frac{y' - u}{x} \\
\frac{1}{x} &= \frac{1}{y' - u}u' \\
&= \frac{1}{\frac{4u - 2}{1 + u} - u}u' \\
&= \frac{1 + u}{4u - 2 - u - u^2}u' \\
&= \frac{-1 - u}{u^2 - 3u + 2}u' \\
&= \frac{-1 - u}{(u - 1)(u - 2)}u' \\
&= \left(\frac{2}{u - 1} - \frac{3}{u - 2}\right)u'
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{x}\,dx &= \int \left(\frac{2}{u - 1} - \frac{3}{u - 2}\right)du \\
\log|x| + C_1 &= 2\log|u - 1| - 3Logu - 2 \\
C_1 &= 2\log|u - 1| - 3Logu - 2 - \log|x| \\
&= \log\left|\frac{(u - 1)^2}{(u - 2)^3x}\right| \\
C_2 &= \frac{(u - 1)^2}{(u - 2)^3x} \\
&= \frac{\left(\frac{y}{x} - 1\right)^2}{\left(\frac{y}{x} - 2\right)^3x} \\
&= \frac{(y - x)^2}{(y - 2x)^3}
\end{aligned}$$

$$(y - x)^2 = C(y - 2x)^3$$

(3) $(2x^2+y^2)y' = 2xy$

$$\begin{aligned} y' &= \frac{2xy}{2x^2+y^2} \\ &= \frac{2\dfrac{y}{x}}{2+\left(\dfrac{y}{x}\right)^2} \end{aligned}$$

$u = \dfrac{y}{x}$ とおく

$$\begin{aligned} \frac{1}{x} &= \frac{1}{y'-u}u' \\ &= \frac{1}{\dfrac{2u}{2+u^2}-u}u' \\ &= \frac{2+u^2}{2u-2u-u^3}u' \\ &= \frac{-2-u^2}{u^3}u' \\ &= \left(-\frac{2}{u^3}-\frac{1}{u}\right)u' \end{aligned}$$

$$\begin{aligned} \int \frac{1}{x}\,dx &= \int \left(-\frac{2}{u^3}-\frac{1}{u}\right)du \\ \log|x| + C_1 &= \frac{1}{u^2} - \log|u| \end{aligned}$$

$$\begin{aligned} C_1 &= \frac{1}{u^2} - \log|u| - \log|x| \\ &= \frac{1}{u^2} - \log|ux| \\ &= \frac{x^2}{y^2} - \log|y| \end{aligned}$$

$$x^2 = y^2(\log|y| + C)$$

(4) $xy' = y(\log y - \log x)$

$$y' = \frac{y}{x}\log\frac{y}{x}$$

$u = \dfrac{y}{x}$ とおく

$$\begin{aligned} u' &= \frac{y'-u}{x} \\ \frac{1}{x} &= \frac{1}{u\log u - u}u' \\ &= \frac{1}{u(\log u - 1)}u' \end{aligned}$$

$$\begin{aligned} \int \frac{1}{x}\,dx &= \int \frac{\dfrac{1}{u}}{\log u - 1}\,du \\ \log|x| + C_1 &= \log|\log u - 1| \end{aligned}$$

$$\begin{aligned} C_1 &= \log|\log u - 1| - \log|x| \\ &= \log\left|\frac{\log u - 1}{x}\right| \end{aligned}$$

$$C = \frac{\log u - 1}{x}$$

$$Cx + 1 = \log u$$

$$\begin{aligned} e^{Cx+1} &= u \\ &= \frac{y}{x} \end{aligned}$$

$$y = xe^{Cx+1}$$