8　同次形に帰着できる微分方程式

$y' = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ の形の微分方程式は，適当な置換によって変数分離形または同次形に帰着できる。

8.1　$a_1 : b_1 = a_2 : b_2$ の場合

$u = a_2x+b_2y+c_2$ とおけば変数分離形になる（第 6 回 $y' = f(a_2x+b_2y+c_2)$ の特別の場合）。

例.　$y' = \dfrac{2x-4y-1}{3x-6y-2}$

$$
\begin{aligned}
y' &= \frac{6x-12y-3}{3(3x-6y-2)} \\
&= \frac{2(3x-6y-2)+1}{3(3x-6y-2)}
\end{aligned}
$$

$u = 3x-6y-2$ とおくと

$$
\begin{aligned}
u' &= 3-6y' \\
&= 3-6\cdot\frac{2u+1}{3u} \\
&= \frac{3u-4u-2}{u} \\
&= \frac{-u-2}{u}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ -1 &= \frac{u}{u+2}u' \\
&= \left(1-\frac{2}{u+2}\right)u' \\
\int -1\,dx &= \int\left(1-\frac{2}{u+2}\right)du \\
-x+C_1 &= u-2\log|u+2| \\
&= 3x-6y-2-2\log|3x-6y| \\
\therefore\ C_1+2 &= 4x-6y-2\log|3x-6y| \\
C &= 2x-3y-\log|3x-6y|
\end{aligned}
$$

8.2　$a_1 : b_1 \neq a_2 : b_2$ の場合

$\begin{cases} a_1x+b_1y+c_1 = a_1(x-\alpha)+b_1(y-\beta) \\ a_2x+b_2y+c_2 = a_2(x-\alpha)+b_2(y-\beta) \end{cases}$ となる α, β を見つけて，

$\begin{cases} X = x-\alpha \\ Y = y-\beta \end{cases}$ とおくと

$$Y' = \frac{dY}{dX} = \frac{\frac{dY}{dx}}{\frac{dX}{dx}} = \frac{y'}{1} = y' = f\left(\frac{a_1X+b_1Y}{a_2X+b_2Y}\right)$$

となり，同次形　$Y' = \dfrac{a_1X+b_1Y}{a_2X+b_2Y}$ が得られる。

なお，α, β は，連立方程式 $\begin{cases} a_1x+b_1y+c_1 = 0 \\ a_2x+b_2y+c_2 = 0 \end{cases}$ を解いて得られる。

例. $y' = \dfrac{5x+9y+17}{3x-y-9}$

$$\begin{cases} 5x+9y+17=0 \\ 3x-y-9=0 \end{cases}$$

$$\therefore \begin{cases} x=2 \\ y=-3 \end{cases}$$

$$\therefore \quad y' = \frac{5(x-2)+9(y+3)}{3(x-2)-(y+3)}$$

$\begin{cases} X = x-2 \\ Y = y+3 \end{cases}$ とおく

$$Y' = \frac{5X+9Y}{3X-Y}$$

$U = \dfrac{Y}{X}$ とおくと

$$\begin{aligned} U' &= \frac{Y'-U}{X} \\ \frac{1}{X} &= \frac{1}{Y'-U}U' \\ &= \frac{1}{\frac{5+9U}{3-U}-U}U' \\ &= \frac{3-U}{5+9U-3U+U^2}U' \\ &= \frac{3-U}{U^2+6U+5}U' \\ &= \frac{3-U}{(U+1)(U+5)}U' \\ &= \left(\frac{1}{U+1}-\frac{2}{U+5}\right)U' \end{aligned}$$

$$\begin{aligned} \int \frac{1}{X}\,dX &= \int \left(\frac{1}{U+1}-\frac{2}{U+5}\right)dU \\ \log|X| + C_1 &= \log|U+1| - 2\log|U+5| \end{aligned}$$

$$\begin{aligned} C_1 &= \log|U+1| - 2\log|U+5| - \log|X| \\ &= \log\left|\frac{U+1}{(U+5)^2X}\right| \\ &= \log\left|\frac{Y+X}{(Y+5X)^2}\right| \\ &= \log\left|\frac{y+3+x-2}{(y+3+5x-10)^2}\right| \\ &= \log\left|\frac{x+y+1}{(5x+y-7)^2}\right| \end{aligned}$$

$$\frac{x+y+1}{(5x+y-7)^2} = \pm e^{C_1} = C$$

問題

次の微分方程式を解きなさい。

(1) $y' = \dfrac{10x+7y-1}{x-y+5}$

(2) $y' + \dfrac{6x+3y+2}{4x+2y+1} = 0$

(3) $(5x-y-9)y' + 3x - 7y + 1 = 0$

(4) $y' = \sqrt{\dfrac{2x-y+1}{x-2}}$ （$\sqrt{\ }$がついているが同様にしてできる）

解答

(1) $y' = \dfrac{10x+7y-1}{x-y+5}$

$$\begin{cases} 10x+7y-1=0 \\ x-y+5=0 \end{cases}$$

$\therefore \quad x=-2,\ y=3$

$$\begin{cases} X=x+2 \\ Y=y-3 \end{cases} \quad \text{とおく}$$

$$Y' = \frac{10X+7Y}{X-Y} = \frac{10+7Y/X}{1-Y/X}$$

$U = \dfrac{Y}{X}$ とおく

$$U' = \frac{Y'-U}{X}$$

$$\begin{aligned}
\frac{1}{X} &= \frac{1}{Y'-U}U' \\
&= \frac{1}{\dfrac{10+7U}{1-U}-U}U' \\
&= \frac{1-U}{10+7U-U+U^2}U' \\
&= \frac{1-U}{U^2+6U+10}U' \\
&= \frac{-(U+3)+4}{(U+3)^2+1}U'
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{X}\,dX &= \int \frac{-(U+3)+4}{(U+3)^2+1}\,dU \\
\log|X| + C_1 &= -\frac{1}{2}\log\left|(U+3)^2+1\right| \\
&\quad +4\tan^{-1}(U+3) \\
-2C_1 &= \log\left|U^2+6U+10\right| + 2\log|X| \\
&\quad -8\tan^{-1}(U+3) \\
C &= \log\left|(U^2+6U+10)X^2\right| \\
&\quad -8\tan^{-1}(U+3) \\
&= \log\left|Y^2+6XY+10X^2\right| \\
&\quad -8\tan^{-1}\left(\frac{Y}{X}+3\right) \\
&= \log\left((y-3)^2+6(x+2)(y-3)\right. \\
&\qquad \left.+10(x+2)^2\right) \\
&\quad -8\tan^{-1}\left(\frac{y-3}{x+2}+3\right) \\
&= \log\left((y-3)^2+6(x+2)(y-3)\right. \\
&\qquad \left.+10(x+2)^2\right) \\
&\quad -8\tan^{-1}\left(\frac{y+3x+3}{x+2}\right)
\end{aligned}$$

(2) $y' + \dfrac{6x+3y+2}{4x+2y+1} = 0$

$$\begin{aligned}
y' &= -\frac{12x+6y+4}{2(4x+2y+1)} \\
&= -\frac{3(4x+2y+1)+1}{2(4x+2y+1)}
\end{aligned}$$

$u = 4x+2y+1$ とおく

$$\begin{aligned}
u' &= 4+2y' \\
&= 4-\frac{3u+1}{u} \\
&= \frac{u-1}{u}
\end{aligned}$$

$$\begin{aligned}
1 &= \frac{u}{u-1}u' \\
&= \left(1+\frac{1}{u-1}\right)u' \\
x + C_1 &= u + \log|u-1| \\
&= 4x+2y+1+\log|4x+2y| \\
C_1 - 1 &= 3x+2y+\log|4x+2y| \\
\therefore\ C &= 3x+2y+\log|4x+2y|
\end{aligned}$$

(3) $(5x-y-9)y'+3x-7y+1=0$

$$\begin{aligned} y' &= -\frac{3x-7y+1}{5x-y-9} \\ &= -\frac{3(x-2)-7(y-1)}{5(x-1)-(y-1)} \end{aligned}$$

$X=x-2,\ Y=y-1$ とおく

$$\begin{aligned} Y' &= -\frac{3X-7Y}{5X-Y} \\ &= -\frac{3-7\dfrac{Y}{X}}{5-\dfrac{Y}{X}} \end{aligned}$$

$U=\dfrac{Y}{X}$ とおく

$$\begin{aligned} U' &= \frac{Y'-U}{X} \\ \frac{1}{X} &= \frac{1}{Y'-U}U' \\ &= \frac{1}{-\dfrac{3-7U}{5-U}-U}U' \\ &= \frac{5-U}{-3+7U-5U+U^2}U' \\ &= \frac{5-U}{U^2+2U-3}U' \\ &= \frac{5-U}{(U+3)(U-2)}U' \\ &= \left(\frac{-\dfrac{8}{5}}{U+3}+\frac{\dfrac{3}{5}}{U-2}\right)U' \end{aligned}$$

$$\begin{aligned} \log|X|+C_1 &= -\frac{8}{5}\log|U+3| \\ &\quad +\frac{3}{5}\log|U-2| \\ 5C_1 &= -8\log|U+3|+3\log|U-2| \\ &\quad -5\log|X| \\ &= \log\left|\frac{(U-2)^3}{(U+3)^8X^5}\right| \\ C &= \frac{(U-2)^3X^3}{(U+3)^8X^8} \\ &= \frac{(Y-2X)^3}{(Y+3X)^8} \\ &= \frac{((y-1)-2(x-2))^3}{((y-1)+3(x-2))^8} \\ &= \frac{(y-2x+3)^3}{(y+3x-7)^8} \end{aligned}$$

$$(y-2x+3)^3 = C(y+3x-7)^8$$

(4) $y' = \sqrt{\dfrac{2x-y+1}{x-2}}$

$$y' = \sqrt{\frac{2(x-2)-(y-5)}{x-2}}$$

$X = x-2,\ Y = y-5$ とおく

$$\begin{aligned} Y' &= \sqrt{\frac{2X-Y}{X}} \\ &= \sqrt{2-\frac{Y}{X}} \end{aligned}$$

$U = \dfrac{Y}{X}$ とおく

$$\begin{aligned} \frac{1}{X} &= \frac{1}{Y'-U}U' \\ &= \frac{1}{\sqrt{2-U}-U}U' \end{aligned}$$

$T = \sqrt{2-U}$ とおく

$$U = 2-T^2$$

$$\therefore \quad U' = -2TT'$$

$$\begin{aligned} \frac{1}{X} &= \frac{1}{T-(2-T^2)}(-2T)T' \\ &= \frac{-2T}{T^2+T-2}T' \\ &= \frac{-2T}{(T+2)(T-1)}T' \\ &= \left(\frac{-\frac{4}{3}}{T+2} + \frac{-\frac{2}{3}}{T-1}\right)T' \end{aligned}$$

$$\begin{aligned} \log|X| + C_1 &= -\frac{4}{3}\log|T+2| \\ &\quad -\frac{2}{3}\log|T-1| \end{aligned}$$

$$\begin{aligned} -3C_1 &= 4\log|T+2| + 2\log|T-1| \\ &\quad +3\log|X| \\ &= \log\left|(T+2)^4(T-1)^2X^3\right| \end{aligned}$$

$$\begin{aligned} C &= (T+2)^4(T-1)^2X^3 \\ &= \left(\sqrt{2-U}+2\right)^4\left(\sqrt{2-U}-1\right)^2X^3 \\ &= \left(\sqrt{2-\frac{Y}{X}}+2\right)^4 \\ &\quad \left(\sqrt{2-\frac{Y}{X}}-1\right)^2X^3 \\ &= \left(\sqrt{2X-Y}+2\sqrt{X}\right)^4 \\ &\quad \left(\sqrt{2X-Y}-\sqrt{X}\right)^2 \\ &= \left(\sqrt{4x-y-3}+2\sqrt{x-2}\right)^4 \\ &\quad \left(\sqrt{4x-y-3}-\sqrt{x-2}\right)^2 \end{aligned}$$