# 9 第１回試験

## 問題　次の条件を満たす曲線の方程式を求めよ。

(1) $y' = 4y^2 \sin 2x$ かつ $y\left(\dfrac{\pi}{6}\right) = 2$

(2) $(x-2y)y' = x - y - 1$

(3) $y' = (4x - y - 2)^2$ かつ $(0, 2)$ を通る

(4) $(x^2 + 2xy)y' = y^2 + 2xy$

(5) $(x^2 + 5)y' = xy$ かつ $x = 2$ のとき $y = 12$

(6) 曲線上の任意の点 P における接線の $y$ 切片（$y$ 軸との交点の $y$ 座標）と法線の $x$ 切片（$x$ 軸との交点の $x$ 座標）が等しいような曲線。

## 解答

(1) $y' = 4y^2 \sin 2x$ かつ $y\left(\dfrac{\pi}{6}\right) = 2$

$$\begin{aligned}
\frac{1}{y^2}y' &= 4\sin 2x \\
\int \frac{1}{y^2}\,dy &= \int 4\sin 2x\,dx \\
-\frac{1}{y} &= -2\cos 2x + C \\
y &= \frac{1}{2\cos 2x - C}
\end{aligned}$$

$$\begin{aligned}
2 &= \frac{1}{2\cos\left(\dfrac{\pi}{3}\right) - C} \\
&= \frac{1}{1 - C} \\
\therefore\ C &= \frac{1}{2} \\
y &= \frac{1}{2\cos 2x - \dfrac{1}{2}} \\
&= \frac{2}{4\cos 2x - 1}
\end{aligned}$$

(2) $(x-2y)y' = x - y - 1$

$$\begin{aligned}
y' &= \frac{x - y - 1}{x - 2y} \\
&= \frac{(x-2) - (y-1)}{(x-2) - 2(y-1)}
\end{aligned}$$

$X = x - 2,\ Y = y - 1$ とおく

$$\begin{aligned}
Y' &= \frac{X - Y}{X - 2Y} \\
&= \frac{1 - \dfrac{Y}{X}}{1 - 2\dfrac{Y}{X}}
\end{aligned}$$

$U = \dfrac{Y}{X}$ とおく

$$\begin{aligned}
\frac{1}{X} &= \frac{1}{Y' - U}U' \\
&= \frac{1}{\dfrac{1-U}{1-2U} - U}U' \\
&= \frac{1-2U}{1 - 2U + 2U^2}U'
\end{aligned}$$

$$\begin{aligned}
\log|X| + C_1 &= -\frac{1}{2}\log\left|1 - 2U + 2U^2\right| \\
-2C_1 &= \log\left|1 - 2U + 2U^2\right| + 2\log|X| \\
&= \log\left|(1 - 2U + 2U^2)X^2\right| \\
C_2 &= (1 - 2U + 2U^2)X^2 \\
&= X^2 - 2XY + 2Y^2 \\
&= (x-2)^2 - 2(x-2)(y-1) + 2(y-1)^2 \\
C &= x^2 - 2xy + 2y^2 - 2x
\end{aligned}$$

(3) $y' = (4x-y-2)^2$ かつ $(0,2)$ を通る

$u = 4x - y - 2$ とおく

$$\begin{aligned}
u' &= 4 - y' \\
&= 4 - u^2 \\
-1 &= \frac{1}{u^2-4}u' \\
&= \frac{1}{4}\left(\frac{1}{u-2} - \frac{1}{u+2}\right)u' \\
\int -4\,dx &= \int \left(\frac{1}{u-2} - \frac{1}{u+2}\right)du \\
-4x + C_1 &= \log|u-2| - \log|u+2| \\
&= \log\left|\frac{u-2}{u+2}\right| \\
\pm e^{-4x+C_1} &= \frac{u-2}{u+2}
\end{aligned}$$

$$\begin{aligned}
Ce^{-4x} &= \frac{4x-y-4}{4x-y} \\
&= 1 - \frac{4}{4x-y} \\
4x - y &= \frac{4}{1-Ce^{-4x}} \\
y &= 4x - \frac{4}{1-Ce^{-4x}} \\
2 &= -\frac{4}{1-C} \\
\therefore C &= 3 \\
y &= 4x - \frac{4}{1-3e^{-4x}}
\end{aligned}$$

(4) $(x^2+2xy)y' = y^2 + 2xy$

$$\begin{aligned}
y' &= \frac{y^2+2xy}{x^2+2xy} \\
&= \frac{\left(\frac{y}{x}\right)^2 + 2\frac{y}{x}}{1+2\frac{y}{x}}
\end{aligned}$$

$u = \dfrac{y}{x}$ とおく

$$\begin{aligned}
u' &= \frac{y'-u}{x} \\
\frac{1}{x} &= \frac{1}{\frac{u^2+2u}{1+2u} - u}u' \\
&= \frac{-2u-1}{u^2-u}u' \\
&= \frac{-2u-1}{u(u-1)}u' \\
&= \left(\frac{1}{u} - \frac{3}{u-1}\right)u'
\end{aligned}$$

$$\begin{aligned}
\log|x| + C_1 &= \log|u| - 3\log|u-1| \\
C_1 &= \log|u| - \log\left|(u-1)^3\right| - \log|x| \\
&= \log\left|\frac{u}{(u-1)^3x}\right| \\
C_2 &= \frac{u}{(u-1)^3x} \\
&= \frac{ux^2}{(u-1)^3x^3} \\
&= \frac{xy}{(y-x)^3} \\
xy &= C(y-x)^3
\end{aligned}$$

(5) $(x^2+5)y' = xy$ かつ $x=2$ のとき $y=12$

$$
\begin{aligned}
\frac{1}{y}y' &= \frac{x}{x^2+5} \\
\int \frac{1}{y}\,dy &= \int \frac{x}{x^2+5}\,dx \\
\log|y| &= \frac{1}{2}\log\left|x^2+5\right| + C_1
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \log|y| - \log\left|\sqrt{x^2+5}\right| \\
&= \log\left|\frac{y}{\sqrt{x^2+5}}\right|
\end{aligned}
$$

$$
\begin{aligned}
\frac{y}{\sqrt{x^2+5}} &= \pm e^{C_1} = C \\
y &= C\sqrt{x^2+5} \\
12 &= C\sqrt{4+5} \\
\therefore\ C &= 4 \\
y &= 4\sqrt{x^2+5}
\end{aligned}
$$

(6) 曲線上の任意の点 P における接線の $y$ 切片（$y$ 軸との交点の $y$ 座標）と法線の $x$ 切片（$x$ 軸との交点の $x$ 座標）が等しいような曲線。

P$(a,b)$ における接線の傾きを $m$ とおく

接線 : $y = m(x-a)+b$

$y$ 切片 : $y = -am+b$

法線 : $y = -\frac{1}{m}(x-a)+b$

$x$ 切片 : $x = a+mb$

$\therefore$ $-am+b = a+mb$

$-(a+b)m = a-b$

ゆえに $-(x+y)y' = x-y$

$$
\begin{aligned}
y' &= -\frac{x-y}{x+y} \\
&= -\frac{1-\frac{y}{x}}{1+\frac{y}{x}}
\end{aligned}
$$

$u = \frac{y}{x}$ とおく

$$
\begin{aligned}
\frac{1}{x} &= \frac{1}{y'-u}u' \\
&= \frac{1}{-\frac{1-u}{1+u}-u}u' \\
&= \frac{1+u}{-1-u^2}u' \\
&= -\left(\frac{u}{1+u^2}+\frac{1}{1+u^2}\right)u'
\end{aligned}
$$

$$
\begin{aligned}
\log|x| + C_1 &= -\frac{1}{2}\log\left|1+u^2\right| - \tan^{-1}u \\
-C_1 - \tan^{-1}u &= \log|x| + \frac{1}{2}\log\left|1+u^2\right| \\
&= \log\left|\sqrt{x^2(1+u^2)}\right| \\
\sqrt{x^2(1+u^2)} &= Ce^{-\tan^{-1}u} \\
\sqrt{x^2+y^2} &= Ce^{-\tan^{-1}\frac{y}{x}}
\end{aligned}
$$

極座標で表すと

$$r = Ce^{-\theta}$$