15　高次微分方程式

n 次の微分方程式

$$(y')^n + f_1(x,y)(y')^{n-1} + + \cdots + f_{n-1}(x,y)y' + f_n(x,y) = 0 \ (n \geqq 2)$$

は，1 次の方程式の積に因数分解できれば解くことができる。

例 1　$(y')^2 + (xy - 4x + y + 1)y' + xy^2 - 3xy - 4x = 0$

$$(y')^2 + (xy - 4x + y + 1)y' + x(y-4)(y+1) = 0$$
$$(y' + x(y-4))\,(y' + (y+1)) = 0$$
$$\therefore\ y' + x(y-4) = 0 \qquad \text{または} \qquad y' + (y+1) = 0$$

(i) $y' + x(y-4) = 0$ のとき

$$\frac{y'}{y-4} = -x$$
$$\int \frac{1}{y-4}\,dy = \int x\,dx$$
$$\log|y-4| = \frac{1}{2}x^2 + C_1$$
$$y - 4 = \pm e^{\frac{1}{2}x^2 + C_1} = \pm e^{C_1} e^{\frac{1}{2}x^2} = Ce^{\frac{1}{2}x^2}$$
$$y - Ce^{\frac{1}{2}x^2} - 4 = 0$$

(ii) $y' + (y+1) = 0$ のとき

$$\frac{y'}{y+1} = -1$$
$$\int \frac{1}{y+1}\,dy = \int -1\,dx$$
$$\log|y+1| = -x + C_1$$
$$y + 1 = \pm e^{-x+C_1} = Ce^{-x}$$
$$y - Ce^{-x} + 1 = 0$$

2 つ合わせて

$$\left(y - Ce^{\frac{1}{2}x^2} - 4\right)(y - Ce^{-x} + 1) = 0$$

問題

(2)〜(5) は第 10 回以降（次回試験範囲）の復習問題

(1)　$x(y')^2 + (x + y - x^2y)y' - xy(x+y) = 0$

(2)　$y' + \dfrac{x}{x^2+1}y = x^2y^3$

(3)　$3x^2y^2 - 2x(y-1) + y^3 + ((3x-1)y^2 + 2x^3y - x^2)y' = 0$

(4)　$y = xy' + 2\sqrt{1-y'}$

(5)　$x^2 \sin x - y + xy' = 0$

解答

(1) $x(y')^2+(x+y-x^2y)y'-xy(x+y)=0$

$$(xy'+x+y)\,(y'-xy)=0$$

(i) $xy'+x+y=0$ のとき

$$\begin{aligned} y' &= -\frac{x+y}{x} \\ &= -1-\frac{y}{x} \end{aligned}$$

$u=\dfrac{y}{x}$ とおく

$$\begin{aligned} \frac{1}{x} &= \frac{1}{y'-u}u' \\ &= \frac{1}{-1-2u}u' \end{aligned}$$

$$\begin{aligned} \int \frac{1}{x}\,dx &= \int \frac{-1}{2u+1}\,du \\ \log|x|+C_1 &= -\frac{1}{2}\log|2u+1| \\ -2C_1 &= \log|2u+1|+2\log|x| \\ &= \log\left|(2u+1)x^2\right| \\ &= \log\left|2xy+x^2\right| \end{aligned}$$

$\therefore\ 2xy+x^2=C$

(ii) $y'-xy=0$ のとき

$$y'=xy$$

$$\frac{1}{y}y'=x$$

$$\int \frac{1}{y}\,dy=\int x\,dx$$

$$\log|y|=\frac{1}{2}x^2+C_1$$

$$\begin{aligned} y &= \pm e^{\frac{1}{2}x^2+C_1} \\ &= Ce^{\frac{1}{2}x^2} \end{aligned}$$

ゆえに

$$\left(2xy+x^2-C\right)\left(y-Ce^{\frac{1}{2}x^2}\right)=0$$

(2) $y' + \dfrac{x}{x^2+1}y = x^2y^3$

$$-2y^{-3}y' - \frac{2x}{x^2+1}y^{-2} = -2x^2$$

$w = y^{-2}$ とおく。

$$w' - \frac{2x}{x^2+1}w = -2x^2$$

$$\begin{aligned} F(x) &= -\int \frac{2x}{x^2+1}\,dx \\ &= -\log x^2+1 \\ G(x) &= e^{-\log x^2+1} \\ &= \frac{1}{x^2+1} \end{aligned}$$

$$\begin{aligned} H(x) &= \int \frac{-2x^2}{x^2+1}\,dx \\ &= \int \left(-2 + \frac{2}{x^2+1}\right)dx \\ &= -2x + 2\tan^{-1}x \end{aligned}$$

$$w = (-2x + 2\tan^{-1}x + C)(x^2+1)$$

$$y^2 = \frac{1}{(-2x+2\tan^{-1}x+C)(x^2+1)}$$

(3) $3x^2y^2 - 2x(y-1) + y^3 + ((3x-1)y^2 + 2x^3y - x^2)y' = 0$

$$\begin{aligned} P(x,y) &= 3x^2y^2 - 2x(y-1) + y^3 \\ Q(x,y) &= (3x-1)y^2 + 2x^3y - x^2 \end{aligned}$$

$$P_y - Q_x = (6x^2y - 2x + 3y^2) - (3y^2 + 6x^2y - 2x) = 0$$

$$\begin{aligned} f(x,y) &= \int (3x^2y^2 - 2x(y-1) + y^3)\,dx + A(y) \\ &= x^3y^2 - x^2(y-1) + xy^3 + A(y) \\ &= x^3y^2 - x^2y + x^2 + xy^3 + A(y) \end{aligned}$$

$$\begin{aligned} f(x,y) &= \int ((3x-1)y^2 + 2x^3y - x^2)\,dy + B(x) \\ &= \frac{1}{3}(3x-1)y^3 + x^3y^2 - x^2y + B(x) \\ &= xy^3 - \frac{1}{3}y^3 + x^3y^2 - x^2y + B(x) \end{aligned}$$

$$\therefore\ f(x,y) = x^3y^2 - x^2y + x^2 + xy^3 - \frac{1}{3}y^3$$

ゆえに $x^3y^2 - x^2y + x^2 + xy^3 - \dfrac{1}{3}y^3 = C$

(4) $y = xy' + 2\sqrt{1-y'}$

$y' = t$ とおく。

$$y = xt + 2\sqrt{1-t}$$
$$y' = t + xt' - \frac{t'}{\sqrt{1-t}}$$
$$t'\left(x - \frac{1}{\sqrt{1-t}}\right) = 0$$

(i) $t' = 0$ のとき

$$t = C$$
$$y = Cx + 2\sqrt{1-C} \qquad \cdots \text{一般解}$$

(ii) $x = \dfrac{1}{\sqrt{1-t}}$ のとき

$$\begin{aligned} y &= \frac{t}{\sqrt{1-t}} + 2\sqrt{1-t} \\ &= \frac{2-t}{\sqrt{1-t}} \\ &= \frac{1+1-t}{\sqrt{1-t}} \\ &= \frac{1}{\sqrt{1-t}} + \sqrt{1-t} \end{aligned}$$

$$\therefore\ y = x + \frac{1}{x} \qquad \cdots \text{特異解}$$

(5) $x^2 \sin x - y + xy' = 0$

$$\underbrace{x^2 \sin x - y}_{P} + \underbrace{x}_{Q} = 0$$

$$P_y - Q_x = -1 - 1 = -2$$

$$\frac{P_{-}Q_x}{Q} = -\frac{2}{x} \qquad y \text{ を含まない}$$

$$\begin{aligned} m(x) &= \int -\frac{2}{x}\, dx \\ &= -2\log|x| = \log x^{-2} \end{aligned}$$

$$M(x) = e^{\log x^{-2}} = x^{-2}$$

$$\underbrace{\sin x - \frac{y}{x^2}}_{P} + \underbrace{\frac{1}{x}}_{Q} y' = 0$$

$$\begin{aligned} f(x,y) &= \int \left(\sin x - \frac{y}{x^2}\right) dx + A(y) \\ &= -\cos x + \frac{y}{x} + A(y) \end{aligned}$$

$$\begin{aligned} f(x,y) &= \int \frac{1}{x}\, dy + B(x) \\ &= \frac{y}{x} + B(x) \end{aligned}$$

$$\therefore\ f(x,y) = -\cos x + \frac{y}{x}$$

ゆえに

$$-\cos x + \frac{y}{x} = C$$

$$\therefore\quad y = Cx + x\cos x$$