16 第 2 回試験

問題

(1) $xy' - y = y' - \sqrt{1-x}$

(2) $y' = \dfrac{3x^2 - 4xy - y^2}{2(x^2 + xy - 3y^2)}$

(3) $xy' - y = 4\sqrt{y' - 1}$

(4) $y'^2 - xy' + xy - y^2 + y' - x + y = 0$

(5) $xy' - y = xy^2 \cos x \qquad (x > 0)$

(6) $y' = -\dfrac{4xy - 2y^2 - y\log y}{x + 2xy - y\log y - y}$

解答

(1) $xy' - y = y' - \sqrt{1-x}$

$$(x-1)y' - y = -\sqrt{1-x}$$
$$y' - \frac{1}{x-1}y = \frac{1}{\sqrt{1-x}}$$

$$\begin{aligned} F(x) &= \int -\frac{1}{x-1}\, dx \\ &= -\log(x-1) \end{aligned}$$

$$G(x) = e^{-\log(x-1)} = \frac{1}{x-1}$$

$$\begin{aligned} H(x) &= \int -\frac{1}{(\sqrt{1-x})^3}\, dx \\ &= \frac{2}{\sqrt{1-x}} \end{aligned}$$

$$\begin{aligned} y &= (x-1)\left(\frac{2}{\sqrt{1-x}} + C\right) \\ &= C(x-1) - 2\sqrt{1-x} \end{aligned}$$

(2) $y' = \dfrac{3x^2 - 4xy - y^2}{2(x^2 + xy - 3y^2)}$

$$\underbrace{3x^2 - 4xy - y^2}_{P} + \underbrace{-2(x^2 + xy - 3y\ 2)}_{Q} y' = 0$$

$$P_y - Q_x = (-4x - 2y) + 2(2x + y) = 0$$

$$\begin{aligned} Ff(x,y) &= \int (3x^2 - 4xy - y^2)\, dx + A(y) \\ &= x^3 - 2x^2y - xy^2 + A(y) \end{aligned}$$

$$\begin{aligned} f(x,y) &= \int (-2x^2 - 2xy + 6y^2)\, dy + B(x) \\ &= -2x^2y - xy^2 + 2y^3 \end{aligned}$$

$$\therefore\ f(x,y) = x^3 - 2x^2y - xy^2 + 2y^3 = (x-2y)(x^2-y^2)$$

ゆえに　$(x-2y)(x-y)(x+y) = C$

(3) $xy' - y = 4\sqrt{y'-1}$

$$y = xy' - 4\sqrt{y'-1}$$

$t = y'$ とおく

$$y = xt - 4\sqrt{t-1}$$
$$t = t + xt' - \frac{2}{\sqrt{t-1}}$$
$$t'\left(x - \frac{2}{\sqrt{t-1}}\right) = 0$$

(1) $t' = 0$ のとき

$$t = C$$
$$y = Cx - 4\sqrt{C-1} \quad \cdots \text{一般解}$$

(2) $x = \dfrac{2}{\sqrt{t-1}}$ のとき

$$\begin{aligned} t &= \frac{4}{x^2} + 1 \\ y &= x\left(\frac{4}{x^2} + 1\right) - 4\frac{2}{x} \\ &= x - \frac{4}{x} \quad \cdots \text{特異解} \end{aligned}$$

(4) $y'^2 - xy' + xy - y^2 + y' - x + y = 0$

$$\begin{aligned} 0 &= y'^2 - (x-1)y' + (xy - x + y - y^2) \\ &= y'^2 - (x-1)y' + (x-y)(y-1) \\ &= (y' - x + y)(y' - y + 1) \end{aligned}$$

(i) $y' - x + y = 0$ のとき

$y' + y = x$

$$\begin{aligned} F(x) &= \int 1\,dx = x \\ G(x) &= e^x \\ H(x) &= \int xe^x\,dx \\ &= x(e^x) - \int 1(e^x)\,dx \\ &= xe^x - e^x \end{aligned}$$

$$\begin{aligned} y &= \frac{(x-1)e^x + C}{e^x} \\ &= x - 1 + Ce^{-x} \end{aligned}$$

(ii) $y' - y + 1 = 0$ のとき

$y' = y - 1$

$1 = \dfrac{1}{y-1}y'$

$$\begin{aligned} \int 1\,dx &= \int \frac{1}{y-1}\,dy \\ x + C_1 &= \log|y-1| \end{aligned}$$

$y = Ce^x + 1$

(5) $xy' - y = xy^2\cos x \qquad (x > 0)$

$y' - \dfrac{1}{x}y = (\cos x)y^2$

$-y^{-2}$ をかける

$-y^{-2}y' + \dfrac{1}{x}y^{-1} = -\cos x$

$w = y^{-1}$ とおく

$w' + \dfrac{1}{x}w = -\cos x$

$$F(x) = \int \frac{1}{x}\,dx \quad = \log x$$

$$G(x) = e^{\log x} \quad = x$$

$$\begin{aligned} H(x) &= \int \underbrace{x}_{u}\underbrace{(-\cos x)}_{v'}\,dx \\ &= x(\sin x) - \int 1\sin x\,dx \\ &= x\sin x + \cos x \end{aligned}$$

$$\begin{aligned} w &= \frac{x\sin x + \cos x + C}{x} \\ y &= \frac{x}{x\sin x + \cos x + C} \end{aligned}$$

(6) $y' = -\dfrac{4xy - 2y^2 - y\log y}{x + 2xy - y\log y - y}$

$$\underbrace{4xy - 2y^2 - y\log y}_{P} + (\underbrace{-x - 2xy + y\log y + y}_{Q})y' = 0$$

$$\begin{aligned} P_y - Q_x &= (4x - 4y - \log y - 1) - (-1 - 2y) \\ &= 4x - 2y - \log y \end{aligned}$$

$$\frac{P_y - Q_x}{P} = \frac{4x - 2y - \log y}{y(4x - 2y - \log y)} = \frac{1}{y} \quad \cdots x \text{ を含まない}$$

$$\begin{aligned} n(y) &= -\int \frac{1}{y}\, dy &&= -\log y \\ N(y) &= e^{-\log y} &&= \frac{1}{y} \end{aligned}$$

$$\underbrace{4x - 2y - \log y}_{P} + \underbrace{\left(-\frac{x}{y} - 2x + \log y + 1\right)}_{Q} y' = 0$$

$$f(x, y) = \int 4x - 2y - \log y\, dx + A(y) = 2x^2 - 2xy - x\log y + A(y)$$

$$f(x, y) = \int \left(-\frac{x}{y} - 2x + \log y + 1\right) dy + B(x) = -x\log y - 2xy + y\log y + B(x)$$

$$\therefore\ f(x, y) = -x\log y - 2xy + 2x^2 + y\log y = (x - y)(2x - \log y)$$

ゆえに　$(x - y)(2x - \log y) = C$