19 非同次線形微分方程式

19.1 一般論

非同次線形微分方程式

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_2(x)y'' + p_1(x)y' + p_0(x)y = r(x) \tag{19.1}$$

の一般解は，右辺を 0 で置き換えた同次方程式

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0 \tag{19.2}$$

の一般解 $C_1f_1(x) + C_2f_2(x) + \cdots + C_nf_n(x)$ および (19.1) の特殊解 $g(x)$ の和である。

19.2 定数係数二階の場合

非同次微分方程式 $ay'' + by' + c = r(x)$ の解

特性方程式 $at^2 + bt + c = 0$ の解で場合分け

(1) $t = \alpha$ (2 重解) のとき

$$f(x) = (C_1 + C_2x)e^{\alpha x}$$

$$g(x) = \frac{e^{\alpha x}}{a}\int\left(\int e^{-\alpha x}r(x)\,dx\right)dx$$

(2) $t = \alpha, \beta$ (2 実数解) のとき

$$f(x) = C_1e^{\alpha x} + C_2e^{\beta x}$$

$$g(x) = \frac{1}{a(\alpha-\beta)}\left(e^{\alpha x}\int e^{-\alpha x}r(x)\,dx - e^{\beta x}\int e^{-\beta x}r(x)\,dx\right)$$

(3) $t = p \pm qi$ (2 虚数解) のとき

$$f(x) = e^{px}\left(C_1\sin qx + C_2\cos qx\right)$$

$$g(x) = \frac{e^{px}}{aq}\left(\sin qx\int e^{-px}\cos qx\;r(x)\,dx - \cos qx\int e^{-px}\sin qx\;r(x)\,dx\right)$$

例 $y'' - 2y' + 5y = e^{2x}$

$$t^2 - 2t + 5 = 0$$
$$t = 1 \pm 2i$$

$$\begin{aligned} f(x) &= e^x \left(C_1 \sin 2x + C_2 \cos 2x\right) \\ g(x) &= \frac{e^x}{2}\left(\sin 2x \int e^{-x} \cos 2x \; e^{2x} \; dx - \cos 2x \int e^{-x} \sin 2x \; e^{2x} \; dx\right) \\ &= \frac{e^x}{2}\left(\sin 2x \int e^{x} \cos 2x \; dx - \cos 2x \int e^{x} \sin 2x \; dx\right) \end{aligned}$$

ところで

$$\begin{aligned} \int \underbrace{e^x}_{u} \underbrace{\cos 2x}_{v'} \, dx &= e^x \cos 2x - \int e^x (-2 \sin 2x) \; dx \\ &= e^x \cos 2x + 2 \int \underbrace{e^x}_{u} \underbrace{\sin 2x}_{v'} \, dx \\ &= e^x \cos 2x + 2 \left(e^x \sin 2x - \int e^x (2 \cos 2x) \; dx\right) \\ &= e^x \cos 2x + 2e^x \sin 2x - 4 \int e^x \cos 2x \; dx \\ &= e^x \left(\cos 2x + 2 \sin 2x\right) - 4F(x) \end{aligned}$$

$$\therefore \; 5 \int e^x \cos 2x \; dx = e^x \left(\cos 2x + 2 \sin 2x\right)$$

$$\therefore \; \int e^x \cos 2x \; dx = \frac{e^x}{5} (\cos 2x + 2 \sin 2x)$$

また $$\begin{aligned} \int e^x \sin 2x \; dx &= \frac{1}{2}\left(\int e^x \cos 2x \; dx - e^x \cos 2x\right) \\ &= \frac{e^x}{10}\left(\cos 2x + 2 \sin 2x - 5 \cos 2x\right) \\ &= \frac{e^x}{5} (\sin 2x - 2 \cos 2x) \end{aligned}$$

ゆえに

$$\begin{aligned} g(x) &= \frac{e^x}{2}\frac{e^x}{5}\left(\sin 2x (\cos 2x + 2 \sin 2x) - \cos 2x (\sin 2x - 2 \cos 2x)\right) \\ &= \frac{e^{2x}}{5} \end{aligned}$$

$$\therefore \; y = e^x \left(C_1 \sin 2x + C_2 \cos 2x\right) + \frac{e^{2x}}{5}$$

問題

(1) $y'' + 2y' - 3y = e^{2x}$

(2) $y'' - 2y' + y = x$

(3) $y'' + 4y = \sin 2x$

解答

(1) $y'' + 2y' - 3y = e^{2x}$

$$t^2 + 2t - 3 = 0$$
$$(t-1)(t+3) = 0$$
$$t = 1, -3$$

$$f(x) = C_1 e^x + C_2 e^{-3x}$$

$$\begin{aligned} g(x) &= \frac{1}{1-(-3)}\left(e^x \int e^{-x} e^{2x}\,dx - e^{-3x}\int e^{3x}e^{2x}\,dx\right) \\ &= \frac{1}{4}\left(e^x \int e^x\,dx - e^{-3x}\int e^{5x}\,dx\right) \\ &= \frac{1}{4}\left(e^x e^x - e^{-3}\frac{1}{5}e^{5x}\right) \\ &= \frac{1}{5}e^{2x} \end{aligned}$$

$$\therefore\ y = C_1 e^x + C_2 e^{-3x} + \frac{1}{5}e^{2x}$$

(2) $y'' + 4y' + 4y = 6$

$$t^2 - 2t + 1 = 0$$
$$(t-1)^2 = 0$$
$$t = 1 \quad (\text{重解})$$

$$f(x) = (C_1 + C_2 x)e^x$$

$$\begin{aligned} g(x) &= e^x \int \left(\int \underbrace{e^{-x}}_{u'}\underbrace{x}_{v}\,dx\right) dx \\ &= e^x \int \left(-e^{-x}x - \int -e^{-x}\,dx\right) dx \\ &= e^x \int \underbrace{-e^{-x}}_{u'}\underbrace{(x+1)}_{v}\,dx \\ &= e^x\left(e^{-x}(x+1) - \int e^{-x}\,dx\right) \\ &= e^x e^{-x}(x+2) \\ &= x+2 \end{aligned}$$

$$\therefore\ y = (C_1 + C_2 x)e^x + x + 2$$

(3) $y'' + 4y = \sin 2x$

$$t^2 + 4 = 0$$
$$t = \pm 2i$$

$$\begin{aligned}
f(x) &= e^0(C_1 \sin 2x + C_2 \cos 2x) \\
&= C_1 \sin 2x + C_2 \cos 2x \\
g(x) &= \frac{1}{2}\left(\sin 2x \int \cos 2x \sin 2x \, dx - \cos 2x \int \sin 2x \sin 2x \, dx\right) \\
&= \frac{1}{2}\left(\sin 2x \int \frac{\sin 4x}{2} \, dx - \cos 2x \int \frac{1 - \cos 4x}{2} \, dx\right) \\
&= \frac{1}{2}\left(\sin 2x \frac{-\cos 4x}{8} - \cos 2x \left(\frac{x}{2} - \frac{\sin 4x}{8}\right)\right) \\
&= \frac{1}{16}\big(\underbrace{\sin 4x \cos 2x - \cos 4x \sin 2x}_{\sin(4x-2x)} - 4x \cos 2x\big) \\
&= \frac{1}{16}(\sin 2x - 4x \cos 2x) \\
\therefore \ y &= C_1 \sin 2x + C_2 \cos 2x + \frac{1}{16}(\sin 2x - 4x \cos 2x) \\
&= C_3 \sin 2x + C_2 \cos 2x - \frac{x}{4} \cos 2x
\end{aligned}$$