## 24 第 4 回試験

### 問題

(1) $y' = 2x\sqrt{1-y^2}$

(2) $y' = \dfrac{5y+3x}{y+3x}$

(3) $y' = \dfrac{2x^3 - xy^2 - 2xy}{x^2y + x^2 - 3y^2}$

(4) $y' = 4x + 2y$

(5) $y' = 2y\tan x + 4\sqrt{y} \quad \left(0 < x < \dfrac{\pi}{2}\right)$

(6) $y'' = 4y' - 4y$

(7) $y'' = 2y' + 4x$

(8) $y'' = \dfrac{y'}{x} - \dfrac{3y}{x^2} + \dfrac{9}{\sqrt{x}}$

## 解答

(1) $y' = 2x\sqrt{1-y^2}$

$$\frac{1}{\sqrt{1-y^2}}y' = 2x$$

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \int 2x\,dx$$

$$\sin^{-1} y = x^2 + C$$

$$y = \sin\left(x^2 + C\right)$$

(2) $y' = \dfrac{5y+3x}{y+3x}$

$u = \dfrac{y}{x}$ とおく

$$\begin{aligned} u' &= \frac{y'-u}{x} \\ &= \frac{\dfrac{5u+3}{u+3} - u}{x} \\ &= \frac{\dfrac{-u^2+2u+3}{u+3}}{x} \end{aligned}$$

$$\begin{aligned} -\frac{1}{x} &= \frac{u+3}{u^2-2u-3}u' \\ &= \left(\frac{A}{u-3} + \frac{B}{u+1}\right)u' \\ &= \frac{1}{2}\left(\frac{3}{u-3} - \frac{1}{u+1}\right)u' \end{aligned}$$

$$-\int \frac{2}{x}\,dx = \int \left(\frac{3}{u-3} - \frac{1}{u+1}\right)du$$

$$-2\log|x| + C_1 = 3\log|u-3| - \log|u+1|$$

$$\begin{aligned} C_1 &= 2Logx + 3\log|u-3| - \log|u+1| \\ &= \log\left|\frac{x^2(u-3)^3}{u+1}\right| \\ &= \log\left|\frac{x^3(u-3)^3}{x(u+1)}\right| \\ &= \log\left|\frac{(y-3x)^3}{y+x}\right| \end{aligned}$$

$$\therefore\ \frac{(y-3x)^3}{y+x} = \pm e^{C_1} = C$$

$$(y-3x)^3 = C(y+x)$$

(3) $y' = \dfrac{2x^3 - xy^2 - 2xy}{x^2y + x^2 - 3y^2}$

$$\underbrace{2x^3 - xy^2 - 2xy}_{P} + \underbrace{(-x^2y - x^2 + 3y^2)}_{Q} y' = 0$$

$$P_y - Q_x = (-2xy - 2x) - (-2xy - 2x) = 0$$

$$\begin{aligned} f(x,y) &= \int (2x^3 - xy^2 - 2xy)\,dx + A(y) \\ &= \frac{1}{2}x^4 - \frac{1}{2}x^2y^2 - x^2y + A(y) \end{aligned}$$

$$\begin{aligned} f(x,y) &= \int (-x^2y - x^2 + 3y^2)\,dy + B(x) \\ &= -\frac{1}{2}x^2y^2 - x^2y + y^3 + B(x) \end{aligned}$$

$$f(x,y) = -\frac{1}{2}x^2y^2 - x^2y + \frac{1}{2}x^4 + y^3$$

$$-\frac{1}{2}x^2y^2 - x^2y + \frac{1}{2}x^4 + y^3 = C_1$$

$$x^2y^2 + 2x^2y - x^4 - 2y^3 = C$$

(4) $y' = 4x + 2y$

$$y' - 2y = 4x$$

$$\begin{aligned} F(x) &= \int -2\,dx \\ &= -2x \\ G(x) &= e^{-2x} \\ H(x) &= \int 4xe^{-2x}\,dx \\ &= \int x(4e^{-2x})\,dx \\ &= x(-2e^{-2x}) - \int 1(-2e^{-2x})\,dx \\ &= -2xe^{-2x} - e^{-2x} \end{aligned}$$

$$\begin{aligned} y &= \frac{(-2x-1)e^{-2x} + C}{e^{-2x}} \\ &= -2x - 1 + Ce^{2x} \end{aligned}$$

(5) $y' = 2y\tan x + 4\sqrt{y} \quad \left(0 < x < \dfrac{\pi}{2}\right)$

$$y' - 2(\tan x)y = 4\sqrt{y}$$

$$\frac{1}{2\sqrt{y}}y' - (\tan x)\sqrt{y} = 2$$

$w = \sqrt{y}$ とおく

$$w' - (\tan x)w = 2$$

$$\begin{aligned} F(x) &= \int -\tan x\,dx \\ &= \int \frac{-\sin x}{\cos x}\,dx \\ &= \log(\cos x) \end{aligned}$$

$$\begin{aligned} G(x) &= e^{\log(\cos x)} \\ &= \cos x \\ H(x) &= \int 2\cos x\,dx \\ &= 2\sin x \end{aligned}$$

$$\begin{aligned} w &= \frac{2\sin x + C}{\cos x} \\ &= 2\tan x + \frac{C}{\cos x} \\ y &= \left(2\tan x + \frac{C}{\cos x}\right)^2 \end{aligned}$$

(6) $y'' = 4y' - 4y$

$y'' - 4y' + 4y = 0$
$t^2 - 4t + 4 = 0$

$t = 2$ (重解)
$y = (C_1 + C_2 x)e^{2x}$

(7) $y'' = 2y' + 4x$

$y'' - 2y' = 4x$

$r(x) = 4x$ は $(1$ 次式$)e^{0x}$ の形

$t^2 - 2t = 0$
$(t-2)t = 0$
$t = 2, 0$
$f(x) = C_1 e^{2x} + C_2$

$g(x)$ は $(1$ 次式$)xe^{0x}$ の形

$g(x) = (A + Bx)x$

$$\begin{aligned} y &= Ax + Bx^2 \\ y' &= A + 2Bx \\ y'' &= 2B \end{aligned}$$

$$\begin{aligned} 4x &= y'' - 2y' \\ &= (2B - 2A) - 4Bx \end{aligned}$$

$\therefore\ B = -1, \quad A = -1$
$g(x) = -x - x^2$

$y = C_1 e^{2x} + C_2 - x - x^2$

(8) $y'' = \dfrac{y'}{x} - \dfrac{3y}{x^2} + \dfrac{9}{\sqrt{x}}$

$x^2 y'' - xy' + 3y = 9x^{\frac{3}{2}}$

$x = e^t$ とおく

$\ddot{y} - 2\dot{y} + 3y = 9e^{\frac{3}{2}t}$

$r(t)$ は $(0$ 次式$)e^{\frac{3}{2}t}$ の形

$u^2 - 2u + 3 = 0$
$u = 1 \pm \sqrt{2}i$
$f(t) = e^t(C_1 \sin\sqrt{2}t + C_2 \cos\sqrt{2}t)$

$g(t)$ は $(0$ 次式$)e^{\frac{3}{2}t}$ の形

$$\begin{aligned} y &= Ae^{\frac{3}{2}t} \\ \dot{y} &= \frac{3}{2}Ae^{\frac{3}{2}t} \\ \ddot{y} &= \frac{9}{4}Ae^{\frac{3}{2}t} \end{aligned}$$

$$9e^{\frac{3}{2}t} = \ddot{y} - 2\dot{y} + 3y = \frac{9}{4}Ae^{\frac{3}{2}t}$$

$\therefore\ A = 4$

$g(t) = 4e^{\frac{3}{2}t}$

$$\begin{aligned} y &= e^t(C_1 \sin\sqrt{2}t + C_2 \cos\sqrt{2}t) + 4e^{\frac{3}{2}t} \\ &= x\left(C_1 \sin(\sqrt{2}\log x) + C_2 \cos(\sqrt{2}\log x)\right) + 4x^{\frac{3}{2}} \end{aligned}$$