9.5 加法定理

加法定理

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

倍角公式

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \qquad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

半角公式

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \qquad \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \qquad \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$$

問題 9.5 次の値を求めなさい。

(1) $\sin\frac{5\pi}{12}$　(2) $\tan\frac{7\pi}{8}$　(3) $\cos 2\theta$ $\left(\text{ただし}\quad \sin\theta = \frac{4}{5}\right)$

問題 9.6 2 直線 $y = \frac{1}{3}x$ と $y = 2x$ のなす角を求めなさい。

ヒント 2 直線が x 軸となす角を，それぞれ α,βとおく。

問題 9.7 $0 \leqq \theta < 2\pi$ の範囲で，次の不等式を解きなさい。

(1) $2\cos^2\theta - 1 > \sin\theta$　(2) $2\sin 2\theta + 1 > 2(\sin\theta + \cos\theta)$

積和変換公式

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin\left(\alpha+\beta\right)+\sin\left(\alpha-\beta\right)\right)$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin\left(\alpha+\beta\right)-\sin\left(\alpha-\beta\right)\right)$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos\left(\alpha+\beta\right)+\cos\left(\alpha-\beta\right)\right)$$

$$\sin\alpha\sin\beta = -\frac{1}{2}\left(\cos\left(\alpha+\beta\right)-\cos\left(\alpha-\beta\right)\right)$$

問題 9.8 積を和の形にしなさい。

(1) $\sin 5x\sin 3x$ (2) $\cos\left(x+\frac{\pi}{12}\right)\sin\left(x-\frac{\pi}{12}\right)$

合成

$$a\sin\theta + b\cos\theta \;=\; r\sin(\theta+\alpha) \qquad a\cos\theta + b\sin\theta \;=\; r\cos(\theta-\alpha)$$

ただし, $r=\sqrt{a^2+b^2},\quad \cos\alpha=\frac{a}{r},\quad \sin\alpha=\frac{b}{r}$

問題 9.9 次の式を $r\sin\left(\theta+\alpha\right)$ の形に書き換えなさい。

(1) $3\sin\theta+\sqrt{3}\cos\theta$ (2) $2\cos\theta-2\sin\theta$

9.6 解答

問題 9.5

(1)
$$\begin{aligned}\sin\frac{5\pi}{12} &= \sin 75^\circ = \sin(45+30)^\circ = \sin 45^\circ\cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

(2)
$$\begin{aligned}\tan\frac{7\pi}{8} &= -\tan\frac{\pi}{8} = -\tan 22.5^\circ = -\tan\frac{45}{2}^\circ = -\frac{1-\cos 45^\circ}{1+\cos 45^\circ} \\ &= -\frac{1-\dfrac{1}{\sqrt{2}}}{1+\dfrac{1}{\sqrt{2}}} = -\frac{\sqrt{2}-1}{\sqrt{2}+1} = -\frac{\left(\sqrt{2}-1\right)^2}{2-1} = -\left(3-2\sqrt{2}\right)\end{aligned}$$

(3)
$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\cdot\frac{16}{25} = -\frac{7}{25}$$

問題 9.6

$y = \frac{1}{3}x$ と x 軸のなす角をαとおくと

$$\tan\alpha = \frac{1}{3}$$

$y = 2x$ と x 軸のなす角をβとおくと

$$\tan\beta = 2$$

2 直線のなす角をθとおくと

$$\theta = \beta - \alpha$$

$$\tan\theta = \tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\alpha\tan\beta} = \frac{2-\dfrac{1}{3}}{1+2\cdot\dfrac{1}{3}} = 1$$

$\therefore\quad \theta = 45^\circ$

問題 9.7

(1)
$$2\cos^2\theta - 1 > \sin\theta$$
$$2(1-\sin^2\theta) - 1 - \sin\theta > 0$$
$$-2\sin^2\theta - \sin\theta + 1 > 0$$
$$2\sin^2\theta + \sin\theta - 1 < 0$$
$$(2\sin\theta - 1)(\sin\theta + 1) < 0$$
$$-1 < \sin\theta < \frac{1}{2}$$
$$\therefore\ 0 \leqq \theta < \frac{\pi}{6},\ \frac{5\pi}{6} < \theta < \frac{3\pi}{2},\ \frac{3\pi}{2} < \theta < 2\pi$$

(2)

$$2\sin 2\theta + 1 > 2(\sin\theta + \cos\theta)$$

$$4\sin\theta\cos\theta - 2\sin\theta - 2\cos\theta + 1 > 0$$

$$(2\sin\theta - 1)(2\cos\theta - 1) > 0$$

$$\begin{cases} \sin\theta > \dfrac{1}{2} \\ \cos\theta > \dfrac{1}{2} \end{cases}, \begin{cases} \sin\theta < \dfrac{1}{2} \\ \cos\theta < \dfrac{1}{2} \end{cases}$$

$$\therefore\ \frac{\pi}{6} < \theta < \frac{\pi}{3}\ ,\ \frac{5\pi}{6} < \theta < \frac{5\pi}{3}$$

問題 9.8

(1)

$$\sin 5x \sin 3x = -\frac{1}{2}(\cos 8x - \cos 2x) = -\frac{1}{2}\cos 8x + \frac{1}{2}\cos 2x$$

(2)

$$\cos\left(x + \frac{\pi}{12}\right)\sin\left(x - \frac{\pi}{12}\right) = \sin 2x - \sin\frac{\pi}{6} = \sin 2x - \frac{1}{2}$$

問題 9.9

(1)

$$3\sin\theta + \sqrt{3}\cos\theta = \sqrt{12}\sin(\theta + \alpha) = 2\sqrt{3}\sin(\theta + \alpha)$$

ただし, $\begin{cases} \cos\alpha = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2} \\ \sin\alpha = \dfrac{\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{2} \end{cases}$

ゆえに, $\alpha = \dfrac{\pi}{6}$

$$\therefore \quad 3\sin\theta + \sqrt{3}\cos\theta = 2\sqrt{3}\sin\left(\theta + \frac{\pi}{6}\right)$$

(2)

$$2\cos\theta - 2\sin\theta = \sqrt{8}\sin(\theta + \alpha) = 2\sqrt{2}\sin(\theta + \alpha)$$

ただし, $\begin{cases} \cos\alpha = \dfrac{-1}{\sqrt{2}} \\ \sin\alpha = \dfrac{1}{\sqrt{2}} \end{cases}$

ゆえに, $\alpha = \dfrac{3\pi}{4}$

$$\therefore\ 2\cos\theta - 2\sin\theta = 2\sqrt{2}\sin\left(\theta + \frac{3\pi}{4}\right)$$