2 の $\sqrt{2}$ 乗

$a > 1$ とします.
指数が有理数のとき，次の大小関係が成り立ちます.

$$p < q \to a^p < a^q$$

無理数 r について, $p < r$ であるすべての有理数 p について $a^p < x$ $r < q$ であるすべての有理数 q について $x < a^q$ を満たす実数 x がただ 1 つ定まります．この x を a^r と定義します.

たとえば，$2^{\sqrt{2}}$ は下のような数になります.

$$\begin{array}{rl}
1< \sqrt{2} <2 & \therefore\quad 2^1< 2^{\sqrt{2}} <2^2 \\
1.4< \sqrt{2} <1.5 & \therefore\quad 2^{\frac{14}{10}}< 2^{\sqrt{2}} <2^{\frac{15}{10}} \\
1.41< \sqrt{2} <1.42 & \therefore\quad 2^{\frac{141}{100}}< 2^{\sqrt{2}} <2^{\frac{142}{100}} \\
 & \vdots
\end{array}$$

この大小関係では，指数の分母が 10 の累乗のため，電卓を使っても，近似値を計算できません.
指数の分母が 2 の累乗になるような大小関係を使って，電卓で近似値を求めてみましょう.
($\sqrt{}$ が何重にもなって見づらいので，$\sqrt{\square}$ を $\surd\,(\square)$ のように書くことにします．)

$$\begin{aligned}
\sqrt{2} \times 2^7 &= \sqrt{2} \times 128 = 181.01\cdots > 181 \\
\therefore\ 2^{\sqrt{2}} > 2^{\frac{181}{2^7}} &= \surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(2^{181}))))))) \\
&= \surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(2^{90}\surd\,(2))))))) \\
&= \surd\,(\surd\,(\surd\,(\surd\,(\surd\,(2^{45}\surd\,(\surd\,(2))))))) \\
&= \surd\,(\surd\,(\surd\,(\surd\,(2^{22}\surd\,(2\ \surd\,(\surd\,(2))))))) \\
&= \surd\,(\surd\,(\surd\,(2^{11}\surd\,(\surd\,(2\ \surd\,(\surd\,(2))))))) \\
&= \surd\,(\surd\,(2^5\surd\,(2\ \surd\,(\surd\,(2\ \surd\,(\surd\,(2))))))) \\
&= \surd\,(2^2\surd\,(2\ \surd\,(2\ \surd\,(\surd\,(2\ \surd\,(\surd\,(2))))))) \\
&= 2\ \surd\,(\surd\,(2\ \surd\,(2\ \surd\,(\surd\,(2\ \surd\,(\surd\,(2)))))))
\end{aligned}$$

内側の括弧から計算しますが，電卓で次のようにボタンを押すと求められます.

$\boxed{2}\boxed{\surd}\boxed{\surd}\boxed{\times}\boxed{2}\boxed{=}\boxed{\surd}\boxed{\surd}\boxed{\times}\boxed{2}\boxed{=}\boxed{\surd}\boxed{\times}\boxed{2}\boxed{=}\boxed{\surd}\boxed{\surd}\boxed{\times}\boxed{2}\boxed{=}\quad 2.664\cdots$

$\therefore\ 2^{\sqrt{2}} > 2.664$

また

$$
\begin{aligned}
\sqrt{2} \times 2^{12} &= \sqrt{2} \times 4096 = 5792.61 \cdots < 5793 \\
\therefore\ 2^{\sqrt{2}} < 2^{\frac{5793}{2^{12}}} &= \surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(2^{5793}))))))))))))) \\
&= \surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(2^{2896}\surd\,(2))))))))))))) \\
&= \surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(2^{181}\surd\,(\surd\,(\surd\,(\surd\,(\surd\,(2))))))))))))) \\
&= 2\ \surd\,(\surd\,(2\ \surd\,(2\ \surd\,(\surd\,(2\ \surd\,(\surd\,(2\ \surd\,(\surd\,(\surd\,(\surd\,(\surd\,(2))))))))))))
\end{aligned}
$$

$$
\boxed{2}\,\boxed{\surd}\,\boxed{\surd}\,\boxed{\surd}\,\boxed{\surd}\,\boxed{\surd}\,\boxed{\times}\,\boxed{2}\,\boxed{=}\,\boxed{\surd}\,\boxed{\surd}\,\boxed{\times}\,\boxed{2}\,\boxed{=}\,\boxed{\surd}\,\boxed{\surd}\,\boxed{\times}\,\boxed{2}\,\boxed{=}\,\boxed{\surd}\,\boxed{\times}\,\boxed{2}\,\boxed{=}\,\boxed{\surd}\,\boxed{\surd}\,\boxed{\times}\,\boxed{2}\,\boxed{=}\quad 2.666\cdots
$$

$$\therefore\ 2^{\sqrt{2}} < 2.667$$

2 つの結果から，小数第 2 位までの近似値がわかりました.

$$2^{\sqrt{2}} = 2.66\cdots \qquad (\text{実際は } 2.665144\cdots)$$