## 1.4　解答

**問題 1**

(1)　$a_k = 1 \quad (1, 1, 1, 1, 1, \cdots)$

$$
\begin{aligned}
f_1(x) &= 1 + 1x + 1x^2 + 1x^3 + 1x^4 + \cdots + 1x^k + \cdots \\
&= \frac{1}{1-x}
\end{aligned}
$$

(2)　$a_k = 2^k \quad (1, 2, 4, 8, 16, \cdots)$

$$
\begin{aligned}
f_2(x) &= 1 + 2x + 4x^2 + 8x^3 + \cdots * 2^k x^k + \cdots \\
&= f_1(2x) \\
&= \frac{1}{1-2x}
\end{aligned}
$$

(3)　$a_k = k + 1 \quad (1, 2, 3, 4, 5, \cdots)$

$$
\begin{aligned}
f_3(x) &= 1 + 2x + 3x^2 + 4x^3 + \cdots + (k+1)x^k + \cdots \\
&= f_1'(x) \quad (\text{微分}) \\
&= \frac{1}{(1-x)^2}
\end{aligned}
$$

別解

$$
\begin{aligned}
f_3(x) &= 1 + 2x + 3x^2 + 4x^3 + \cdots + (k+1)x^k + \cdots \\
&= 1 + 1x + 1x^2 + 1x^3 + \cdots + 1x^k + \cdots \\
&\qquad + 1x + 2x^2 + 3x^3 + \cdots + kx^k + \cdots \\
&= f_1(x) + x f_3(x)
\end{aligned}
$$

$$
\begin{aligned}
(1-x)f_3(x) = f_1(x) = \frac{1}{1-x} \\
\therefore\ f_3(x) = \frac{1}{(1-x)^2}
\end{aligned}
$$

(4)　$a_k = 2k + 1 \quad (1, 3, 5, 7, 9, \cdots)$

$$
\begin{aligned}
f_4(x) &= 1 + 3x + 5x^2 + 7x^3 + \cdots + (2k+1)x^k + \cdots \\
&= (2 + 4x + 6x^2 + 8x^3 + \cdots + (2k+2)x^k + \cdots) \\
&\quad - (1 + 1x + 1x^2 + 1x^3 + \cdots + 1x^k + \cdots) \\
&= 2f_3(x) - f_1(x) \\
&= \frac{2}{(1-x)^2} - \frac{1}{1-x} \\
&= \frac{1+x}{(1-x)^2}
\end{aligned}
$$

**問題 2** 次の関数の係数列 (の一般項) を求めなさい。

(5) $f_5(x) = \dfrac{1}{1-3x}$

$$\begin{aligned} f_5(x) &= f_1(3x) \\ &= 1 + 3x + (3x)^2 + (3x)^3 + \cdots + (3x)^k + \cdots \\ a_k &= 3^k \end{aligned}$$

(6) $f_6(x) = \dfrac{1}{(1-x)^3}$

$$\begin{aligned} f_6(x) &= \frac{1}{2}\left(\frac{1}{(1-x)^2}\right)' \\ &= \frac{1}{2}(1 + 2x + 3x^2 + 4x^3 + \cdots + (k+1)x^k + \cdots)' \\ &= \frac{1}{2}(2\cdot 1 + 3\cdot 2x + 4\cdot 3x^2 + 5\cdot 4x^3 + \cdots + (k+2)(k+1)x^k + \cdots) \\ a_k &= \frac{(k+2)(k+1)}{2} \end{aligned}$$

別解

$$\begin{aligned} f_6(x) &= \frac{1}{(1-x)^2}\frac{1}{1-x} \\ &= \left(1 + 2x + 3x^2 + 4x^3 + \cdots + (k+1)x^k + \cdots\right) \\ &\times \left(1 + 1x + 1x^2 + 1x^3 + \cdots + 1x^k + \cdots\right) \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots + (k+1)x^k + \cdots \\ &\qquad + 1x + 2x^2 + 3x^3 + \cdots + kx^k + \cdots \\ &\qquad\qquad + 1x^2 + 2x^3 + \cdots + (k-1)x^k + \cdots \\ &\qquad\qquad\qquad + 1x^3 + \cdots + (k-2)x^k + \cdots \\ &\qquad\qquad\qquad\qquad \vdots \\ &\qquad\qquad\qquad\qquad\qquad + 1x^k + \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_k &= (k+1) + k + \cdots + 2 + 1 \\ &= \frac{1}{2}(k+2)(k+1) \end{aligned}$$

(7) $f_7(x) = \dfrac{1}{1-3x+2x^2}$

$$\begin{aligned} f_7(x) &= \frac{1}{(1-x)(1-2x)} \\ &= \frac{2(1-x)-(1-2x)}{(1-x)(1-2x)} \\ &= \frac{2}{1-2x} - \frac{1}{1-x} \\ &= 2(1 + 2x + 2^2x^2 + 2^3x^3 + \cdots + 2^kx^k + \cdots) \\ &\quad - (1 + 1x + 1x^2 + 1x^3 + \cdots + 1x^k + \cdots) \\ a_k &= 2^{k+1} - 1 \end{aligned}$$

**問題 3** 次の関数の係数列を求めなさい。

(8) $f_8(x) = \dfrac{1}{(1+x)^2}$

$$f_8(x) = (1+x)^{-2} = \sum_{k=0}^{\infty} {}_{-2}\mathrm{C}_k\, x^k$$

$$\begin{aligned} a_k &= {}_{-2}\mathrm{C}_k \\ &= \frac{-2}{1}\frac{-3}{2}\frac{-4}{3}\cdots\cdots\frac{-k}{k-1}\frac{-(k+1)}{k} \\ &= (-1)^k(k+1) \end{aligned}$$

(9) $f_9(x) = \dfrac{1}{(1-x)^3}$

$$f_9(x) = (1-x)^{-3} = \sum_{k=0}^{\infty} {}_{-3}\mathrm{C}_k(-x)^k$$

$$\begin{aligned} a_k &= (-1)^k\frac{-3}{1}\frac{-4}{2}\frac{-5}{3}\cdots\cdots\frac{-(k+1)}{k-1}\frac{-(k+2)}{k} \\ &= \frac{(k+1)(k+2)}{1\cdot 2} \end{aligned}$$

一般に，$\dfrac{1}{(1-x)^n}$ の $x^k$ の係数は

$$a_k = \frac{(k+1)(k+2)\cdots(k+n-1)}{1\cdot 2\cdot 3\cdots k} = {}_{k+n-1}\mathrm{C}_{n-1}$$

(10) $f_{10}(x) = \dfrac{1}{\sqrt{1-x}}$

$$f_{10}(x) = (1-x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} {}_{-\frac{1}{2}}\mathrm{C}_k(-x)^k$$

$$\begin{aligned} a_k &= (-1)^k\frac{-\frac{1}{2}}{1}\frac{-\frac{3}{2}}{2}\frac{-\frac{5}{2}}{3}\cdots\cdots\frac{\frac{2k-1}{2}}{k} \\ &= \frac{1\cdot 3\cdot 5\cdots\cdots(2k-1)}{2\cdot 4\cdot 6\cdots\cdots(2k)} \\ &= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots\cdots(2k-1)(2k)}{\left(2\cdot 4\cdot 6\cdots\cdots(2k)\right)^2} \\ &= \frac{(2k)!}{2^{2k}(k!)^2} \\ &= \frac{{}_{2k}\mathrm{C}_k}{2^{2k}} \end{aligned}$$

**問題 4** 次の漸化式で定義される数列の一般項を，母関数を用いて求めなさい．

(1) $\begin{cases} a_0 = 1 \\ a_k = 2a_{k-1} + k + 1 \end{cases}$

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots \\ &= 1 + (2a_0 + 2)x + (2a_1 + 3)x^2 + \cdots + (2a_{k-1} + k + 1)x^k + \cdots \\ &= (1 + 2x + 3x^2 + 4x^3 + \cdots + (k+1)x^k + \cdots) \\ &\qquad + 2x(a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1}x^{k-1} + \cdots) \\ &= \frac{1}{(1-x)^2} + 2xf(x) \end{aligned}$$

$$(1-2x)f(x) = \frac{1}{(1-x)^2}$$

$$\begin{aligned} f(x) &= \frac{1}{(1-2x)(1-x)^2} = \frac{4}{1-2x} - \frac{2}{1-x} - \frac{1}{(1-x)^2} \\ &= \quad 4(1 + 2x + \ 4x^2 + \cdots + 2^k x^k + \cdots) \\ &\quad - 2(1 + 1x + \ 1x^2 + \cdots + 1x^k + \cdots) \\ &\quad - 1(1 + 2x + \ 3x^2 + \cdots + (k+1)x^k + \cdots) \\ &= \qquad 1 + 4x + 11x^2 + \cdots + \underbrace{(4 \cdot 2^k - k - 3)}_{a_k} x^k + \cdots \end{aligned}$$

$$a_k = 2^{k+2} - k - 3$$

(2) $\begin{cases} a_0 = 1 \\ a_k = 3a_{k-1} + 2^k \end{cases}$

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_k x^k + \cdots \\ &= 1 + (3a_0 + 2)x + (3a_1 + 4)x^2 + (3a_2 x + 8)x^3 + \cdots + (3a_{k-1} + 2^k)x^k + \cdots \\ &= (1 + 2x + 4x^2 + 8x^3 + \cdots + 2^k x^k + \cdots) \\ &\quad + 3(a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_{k-1} x^k * \cdots) \\ &= \frac{1}{1-2x} + 3xf(x) \end{aligned}$$

$$(1-3x)f(x) = \frac{1}{1-2x}$$

$$\begin{aligned} f(x) &= \frac{1}{(1-2x)(1-3x)} = \frac{3}{1-3x} - \frac{2}{1-2x} \\ &= \quad 3(1 + 3x + 3^2 x^2 + 3^3 x^3 + \cdots + 3^k x^k + \cdots) \\ &\quad - 2(1 + 2x + 2^2 x^2 + 2^3 x^3 + \cdots + 2^k x^k + \cdots) \end{aligned}$$

$$a_k = 3^{k+1} - 2^{k+1}$$

(3) $\begin{cases} a_0 = 4 \\ a_1 = 1 \\ a_k = a_{k-1} + 2a_{k-2} \end{cases}$

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_k x^k + \cdots \\ &= 4 + 1x + (a_1 + 2a_0)x^2 + (a_2 + 2a_1)x^3 + (a_3 + 2a_2)x^4 + \cdots + (a_{k+1} + 2a_{k+2})x^k + \cdots \\ &= 4 + x + (a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots + a_{k+1} x^k + \cdots) \\ &\qquad + 2(a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots + a_{k+2} x^k + \cdots) \\ &= 4 + x + x(f(x) - 4) + 2x^2 f(x) \end{aligned}$$

$$(1 - x - 2x^2) f(x) = 4 - 3x$$

$$\begin{aligned} f(x) &= \frac{4 - 3x}{(1 - 2x)(1 + x)} = \frac{\frac{5}{3}}{1 - 2x} + \frac{\frac{7}{3}}{1 + x} \\ &= \quad \frac{5}{3}(1 + 2x + 2^2 x^2 + 2^3 x^3 + \cdots + 2^k x^k + \cdots) \\ &\quad + \frac{7}{3}(1 - x + x^2 - x^3 + \cdots + (-1)^k x^k + \cdots) \\ a_k &= \frac{5 \cdot 2^k + 7 \cdot (-1)^k}{3} \end{aligned}$$

**問題 5** 次の漸化式で定義される数列の一般項を，母関数を用いて求めなさい．

(1) $\begin{cases} a_0 = 3 \\ a_k = a_0 + a_1 + a_2 + \cdots + a_{k-1} \end{cases}$

$$
\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots \\
&= 3 + a_0 x + (a_0 + a_1) x^2 + \cdots + (a_0 + a_1 + a_2 + \cdots + a_{k-1}) x^k + \cdots \\
&= 3 + a_0 x + a_0 x^2 + \cdots + a_0 x^k + \cdots \\
&\qquad + a_1 x^2 + \cdots + a_1 x^k + \cdots \\
&\qquad + \cdots + a_2 x^k + \cdots \\
&\qquad \vdots \\
&\qquad + a_{k-1} x^k + \cdots \\
&\qquad \vdots \\
&= 3 + (1 + x + x^2 + \cdots + x^{k-1} + \cdots) x \\
&\qquad \times (a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-1} x^{k-1} + \cdots) \\
&= 3 + \frac{x}{1-x} f(x)
\end{aligned}
$$

$$\frac{1-2x}{1-x} f(x) = 3$$

$$
\begin{aligned}
f(x) &= \frac{3}{1-2x} - \frac{3x}{1-2x} \\
&= 3 \cdot 1 + 3 \cdot 2x + 3 \cdot 4x^2 + 3 \cdot 8x^3 + \cdots + 3 \cdot 2^k x^k + \cdots \\
&\qquad - 3 \cdot 1x - 3 \cdot 2x^2 - 3 \cdot 4x^3 - \cdots - 3 \cdot 2^{k-1} x^k - \cdots \\
&= 3 + 3 \cdot 1x + 3 \cdot 2x^2 + 3 \cdot 4x^3 + \cdots + 3 \cdot 2^{k-1} x^k + \cdots
\end{aligned}
$$

$\therefore \begin{cases} a_0 = 3 \\ a_k = 3 \cdot 2^{k-1} \ (k \geqq 1) \end{cases}$

(2) $\begin{cases} a_0 = 1 \\ a_1 = 1 \\ a_k = a_{k-2} + 2a_{k-3} + 3a_{k-4} + \cdots + (k-1)a_0 \end{cases}$

$$
\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_k x^k + \cdots \\
&= 1 + 1x + a_0 x^2 + (a_1 + 2a_0)x^3 + (a_2 + 2a_1 + 3a_0)x^4 + \cdots \\
&\qquad \cdots + (a_{k-2} + 2a_{k-3} + 3a_{k-4} + \cdots + (k-1)a_0)x^k + \cdots \\
&= 1 + x + a_0(1x^2 + 2x^3 + 3x^4 + \cdots + (k-1)x^k + \cdots) \\
&\qquad + a_1(1x^3 + 2x^4 + \cdots + (k-2)x^k + \cdots) \\
&\qquad + a_2(1x^4 + \cdots + (k-3)x^k + \cdots) \\
&\qquad \vdots \\
&\qquad + a_{k-2}(x^k + \cdots) \\
&\qquad \vdots \\
&= 1 + x + x^2(1 + 2x + 3x^2 + \cdots + (k+1)x^k + \cdots) \\
&\qquad \times (a_0 + a_1 x + a_2 x^2 + \cdots + a_{k-2}x^{k^2} + \cdots) \\
&= 1 + x + \frac{x^2}{(1-x)^2} f(x)
\end{aligned}
$$

$$
\begin{aligned}
\left(1 - \frac{x^2}{(1-x)^2}\right) f(x) &= 1 + x \\
\frac{1-2x}{(1-x)^2} f(x) &= 1 + x
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \frac{(1+x)(1-x)^2}{1-2x} = \frac{1 - x - x^2 + x^3}{1-2x} \\
&= (1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots + 2^k x^k + \cdots) \\
&\qquad - (1x + 2x^2 + 4x^3 + 8x^4 + \cdots + 2^{k-1}x^k + \cdots) \\
&\qquad - (1x^2 + 2x^3 + 4x^4 + \cdots + 2^{k-2}x^k + \cdots) \\
&\qquad + (1x^3 + 2x^4 + \cdots + 2^{k-3}x^k + \cdots) \\
&= 1 + 1x + 1x^2 + 3x^3 + 6x^4 + \cdots + 3 \cdot 2^{k-3}x^k + \cdots
\end{aligned}
$$

$$
a_k = \begin{cases} 1 & (k = 0, 1, 2) \\ 3 \cdot 2^{k-3} & (k = 3.4.5, \ldots) \end{cases}
$$