1　m 次関数の一表現

整数 n に関する m 次の関数は

$$\begin{aligned} f(n) &= a_0 + a_1(n+1) + a_2(n+1)(n+2) + \cdots + a_m(n+1)(n+2)\cdots(n+m) \\ &= \sum_{k=0}^{m} a_k \cdot {}_{n+k}\mathrm{P}_k \end{aligned}$$

の形で表せる.

問題 1.1 係数 a_k $(k = 0, 1, \ldots, m)$ を求めよ.

$a_k = \dfrac{b_k}{k!}$ とおくと

$$\begin{aligned} f(n) &= \sum_{k=0}^{m} b_k \cdot \frac{{}_{n+k}\mathrm{P}_k}{k!} \\ &= \sum_{k=0}^{m} b_k \cdot {}_{n+k}\mathrm{C}_k \\ &= b_0 + b_1 \cdot {}_{n+1}\mathrm{C}_1 + b_2 \cdot {}_{n+2}\mathrm{C}_2 + \cdots + b_m \cdot {}_{n+m}\mathrm{C}_m \end{aligned}$$

注 1.1 ${}_n\mathrm{C}_k = \begin{cases} 0 & (k < 0, n < k) \\ 1 & (k = 0, k = n) \\ {}_{n-1}\mathrm{C}_{k-1} + {}_{n-1}\mathrm{C}_k & (0 < k < n) \end{cases}$

定義 1.1 関数列 $f_0, f_1, f_2, \ldots, f_m$ を次のように定義する.

$$\begin{cases} f_0(n) = f(n) \\ f_{i+1}(n) = f_i(n+1) - f_i(n) \quad (i = 0, 1, \ldots, m-1) \end{cases}$$

例 1.1

$$\begin{aligned} f_0(n) &= b_0 + b_1(n+1) + b_2 \cdot \frac{1}{2}(n+1)(n+2) + b_3 \cdot \frac{1}{6}(n+1)(n+2)(n+3) + \cdots \\ f_1(n) &= b_1 + b_2(n+2) + b_3 \cdot \frac{1}{2}(n+2)(n+3) + \cdots \\ f_2(n) &= b_2 + b_3(n+3) + \cdots \end{aligned}$$

補題 1.1 $f_i(n) = \displaystyle\sum_{k=i}^{m} b_k \cdot {}_{n+k}\mathrm{C}_{k-i}$

証明 1.1 数学的帰納法で示す.

(i) $i = 0$ について

目標：　$f_0(n) = \displaystyle\sum_{k=0}^{m} b_k \cdot {}_{n+k}\mathrm{C}_k$

$f_0(n) = f(n)$ だから成り立つ.

(ii) $0 \leqq i < m$ について

仮定： $f_i(n) = \sum_{k=i}^{m} b_k \cdot {}_{n+k}\mathrm{C}_{k-i}$ (すべての n について)

目標： $f_{i+1}(n) = \sum_{k=i+1}^{m} b_k \cdot {}_{n+k}\mathrm{C}_{k-i-1}$

$$
\begin{aligned}
f_{i+1}(n) &= f_i(n+1) - f_i(n) \\
&= \sum_{k=i}^{m} b_k \cdot {}_{n+1+k}\mathrm{C}_{k-i} - \sum_{k=i}^{m} b_k \cdot {}_{n+k}\mathrm{C}_{k-i} \\
&= (b_i - b_i) + \sum_{k=i+1}^{m} b_k \left({}_{n+k+1}\mathrm{C}_{k-i} - {}_{n+k}\mathrm{C}_{k-i}\right) \\
&= \sum_{k=i+1}^{m} b_k \cdot {}_{n+k}\mathrm{C}_{k-i-1}
\end{aligned}
$$

成り立つ.

補題 1.2 $f_i(n) = \sum_{j=0}^{i} (-1)^j \ {}_i\mathrm{C}_j \ f(n+i-j)$

証明 1.2 数学的帰納法で示す.

(i) $i = 0$ について
目標: $f_0(n) = f(n)$
成り立つ.

(ii) $0 \leqq i < m$ について

仮定: $f_i(n) = \sum_{j=0}^{i} (-1)^j \ {}_i\mathrm{C}_j \ f(n+i-j)$ (すべての n について)

目標: $f_{i+1}(n) = \sum_{j=0}^{i+1} (-1)^j \ {}_{i+1}\mathrm{C}_j \ f(n+1+i-j)$

$$
\begin{aligned}
f_{i+1}(n) &= f_i(n+1) - f_i(n) \\
&= \sum_{j=0}^{i} (-1)^j \ {}_i\mathrm{C}_j \ f(n+1+i-j) - \sum_{j=0}^{i} (-1)^j \ {}_i\mathrm{C}_j \ f(n+i-j) \\
&= \sum_{j'=0}^{i} (-1)^{j'} \ {}_i\mathrm{C}_{j'} f(n+1+i-j') - \sum_{j'=1}^{i+1} -(-1)^{j'} {}_i\mathrm{C}_{j'-1} f(n+i-j'+1) \\
&= f(n+1+i) + \sum_{j=1}^{i} (-1)^j \left({}_i\mathrm{C}_j + {}_i\mathrm{C}_{j-1}\right) f(n+1+i-j) + (-1)^{i+1} f(n) \\
&= f(n+1+i) + \sum_{j=1}^{i} (-1)^j {}_{i+1}\mathrm{C}_j f(n+1+i-j) + (-1)^{i+1} f(n) \\
&= \sum_{j=0}^{i+1} (-1)^j {}_{i+1}\mathrm{C} \ j f(n+1+i-j)
\end{aligned}
$$

成り立つ.

解答 1.1 補題 1，2 より

$$b_k = f_k(-k-1) = \sum_{j=0}^{k} (-1)^j {}_k\mathrm{C}_j f(-j-1)$$

$$a_k = \frac{\displaystyle\sum_{j=0}^{k} (-1)^j {}_k\mathrm{C}_j f(-j-1)}{k!}$$

例 1.2 $f(n) = n^m$ (最も単純なm次関数) の場合

$$b_k == \sum_{j=0}^{k} (-1)^j {}_k\mathrm{C}_j (-j-1)^m$$

$$a_k = \frac{\displaystyle\sum_{j=0}^{k} (-1)^j {}_k\mathrm{C}_j (-j-1)^m}{k!}$$