**質問**　複素平面の証明を教えてください。

0 でない複素数 $z$，$w$ について

(1)　$\overrightarrow{z} \,/\!/\, \overrightarrow{w} \iff z\overline{w} - \overline{z}\,w = 0$

(2)　$\overrightarrow{z} \perp \overrightarrow{w} \iff z\overline{w} + \overline{z}\,w = 0$

**解答 1**

極表示 $r(\cos\theta + i\sin\theta)$ を $r\,\mathrm{cis}(\theta)$ と略記します．

$z = a\,\mathrm{cis}(\alpha),\quad w = b\,\mathrm{cis}(\beta)$　とする．

$\overline{z} = a\,\mathrm{cis}(-\alpha),\quad \overline{w} = b\,\mathrm{cis}(-\beta)$

(1)

$$\overrightarrow{z} \,/\!/\, \overrightarrow{w} \iff \alpha - \beta = k\pi \ (k \text{ は整数})$$

$$\begin{aligned}
z\overline{w} - \overline{z}\,w = 0 &\iff a\,\mathrm{cis}(\alpha)b\,\mathrm{cis}(-\beta) - a\,\mathrm{cis}(-\alpha)b\,\mathrm{cis}(\beta) = 0 \\
&\iff ab\,\mathrm{cis}(\alpha-\beta) = ab\,\mathrm{cis}(\beta-\alpha) \\
&\iff (\alpha-\beta) - (\beta-\alpha) = 2k\pi \ (k \text{ は整数}) \\
&\iff \alpha - \beta = k\pi \ (k \text{ は整数}) \\
&\iff \overrightarrow{z} \,/\!/\, \overrightarrow{w}
\end{aligned}$$

(2)

$$\overrightarrow{z} \perp \overrightarrow{w} \iff \alpha - \beta = \frac{\pi}{2} + k\pi \ (k \text{ は整数})$$

$$\begin{aligned}
z\overline{w} + \overline{z}\,w = 0 &\iff a\,\mathrm{cis}(\alpha)b\,\mathrm{cis}(-\beta) + a\,\mathrm{cis}(-\alpha)b\,\mathrm{cis}(\beta) = 0 \\
&\iff ab\,\mathrm{cis}(\alpha-\beta) = -ab\,\mathrm{cis}(\beta-\alpha) \\
&\iff (\alpha-\beta) - (\beta-\alpha) = \pi + 2k\pi \ (k \text{ は整数}) \\
&\iff \alpha - \beta = \frac{\pi}{2} + k\pi \ (k \text{ は整数}) \\
&\iff \overrightarrow{z} \perp \overrightarrow{w}
\end{aligned}$$

**解答 2**

(1)

$$\overrightarrow{\boldsymbol{z}} \,/\!/\, \overrightarrow{\boldsymbol{w}} \iff z = aw \ (a \text{ は実数} \fallingdotseq 0)$$

$$\begin{aligned} z\overline{w} - \overline{z}\,w = 0 &\iff \frac{z}{w} = \frac{\overline{z}}{\overline{w}} \\ &\iff \frac{z}{w} = \overline{\left(\frac{z}{w}\right)} \\ &\iff \frac{z}{w} = a \ (a \text{ は実数} \fallingdotseq 0) \\ &\iff z = aw \ (a \text{ は実数} \fallingdotseq 0) \\ &\iff \overrightarrow{\boldsymbol{z}} \,/\!/\, \overrightarrow{\boldsymbol{w}} \end{aligned}$$

(2)

$$\overrightarrow{\boldsymbol{z}} \perp \overrightarrow{\boldsymbol{w}} \iff z = aiw \ (a \text{ は実数} \fallingdotseq 0)$$

$$\begin{aligned} z\overline{w} + \overline{z}\,w = 0 &\iff \frac{z}{w} = -\frac{\overline{z}}{\overline{w}} \\ &\iff \frac{z}{w} = -\overline{\left(\frac{z}{w}\right)} \\ &\iff \frac{z}{w} = ai \ (a \text{ は実数} \fallingdotseq 0) \\ &\iff z = aiw \ (a \text{ は実数} \fallingdotseq 0) \\ &\iff \overrightarrow{\boldsymbol{z}} \perp \overrightarrow{\boldsymbol{w}} \end{aligned}$$