# 1　関数列 $f_n(x) = \displaystyle\int_a^x f_{n-1}(t)\ dt$ の一般項

問題 1.1　関数列 $\{f_n(x)\}$ を

$$
\begin{aligned}
&f_0(x) = f(x)\\
&f_n(x) = \int_a^x f_{n-1}(t)\ dt \quad (n = 1, 2, 3, \dots)
\end{aligned}
$$

と定義する。

$$
f_n(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\ dt \quad (n = 1, 2, 3, \dots)
$$

が成り立つことを示せ。

補題 1.1

$$
\int_a^b \int_a^y y^r g(x)\ dx\ dy = \frac{1}{r+1}\int_a^b \left(b^{r+1} - x^{r+1}\right) g(x)\ dx
$$

証明 **1 (高校生)**

$$
\begin{aligned}
\int_a^b \int_a^y y^r g(x)\ dx\ dy &= \int_a^b \underbrace{y^r}_{u'} \underbrace{\int_a^y g(x)\ dx}_{v}\ dy\\
&= \left[\overbrace{\frac{y^{r+1}}{r+1}}^{u}\overbrace{\int_a^y g(x)\ dx}^{v}\right]_a^b - \int_a^b \overbrace{\frac{y^{r+1}}{r+1}}^{u}\overbrace{g(y)}^{v'}\ dy\\
&= \frac{b^{r+1}}{r+1}\int_a^b g(x)\ dx - \int_a^b \frac{x^{r+1}}{r+1} g(x)\ dx\\
&= \frac{1}{r+1}\int_a^b (b^{r+1} - x^{r+1}) g(x)\ dx
\end{aligned}
$$

証明 **2 (大学生)**

$$
\left\{(x, y)\ \middle|\ a \leqq x \leqq y,\ a \leqq y \leqq b\right\} = \left\{(x, y)\ \middle|\ x \leqq y \leqq b,\ a \leqq x \leqq b\right\}
$$

だから

$$
\begin{aligned}
\int_a^b \int_a^y y^r g(x)\ dx\ dy &= \int_a^b \int_x^b y^r g(x)\ dy\ dx\\
&= \int_a^b \left[\frac{y^{r+1}}{r+1} g(x)\right]_x^b dx\\
&= \frac{1}{r+1}\int_a^b \left(b^{r+1} - x^{r+1}\right) g(x)\ dx
\end{aligned}
$$

補題 1.2

$$\frac{{}_{n-1}\mathrm{C}_k}{n-k} = \frac{{}_{n}\mathrm{C}_k}{n}$$

証明

$$\frac{{}_{n-1}\mathrm{C}_k}{n-k} = \frac{\dfrac{(n-1)!}{(n-1-k)!\,k!}}{n-k} = \frac{(n-1)!}{(n-k)!\,k!} = \frac{\dfrac{n!}{(n-k)!\,k!}}{n} = \frac{{}_{n}\mathrm{C}_k}{n}$$

解答 1.1　$n$ に関する数学的帰納法で示す。

(1)

$$f_1(x) = \int_a^x f_0(t)\,dt = \frac{1}{(1-1)!}\int_a^x (x-t)^{1-1} f(t)\,dt$$

(2)

$$\begin{aligned}
f_{n+1}(x) &= \int_a^x f_n(u)\,du \\
&= \int_a^x \frac{1}{(n-1)!}\int_a^u (u-t)^{n-1} f(t)\,dt\,du \\
&= \frac{1}{(n-1)!}\int_a^x \int_a^u \sum_{k=0}^{n-1} {}_{n-1}\mathrm{C}_k u^{n-1-k}(-t)^k f(t)\,dt\,du \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n-1} {}_{n-1}\mathrm{C}_k \int_a^x \int_a^u u^{\overbrace{n-1-k}^{r}} \overbrace{(-t)^k f(t)}^{g(t)}\,dt\,du \\
&= \frac{1}{(n-1)!}\sum_{k=0}^{n-1} {}_{n-1}\mathrm{C}_k \frac{1}{n-k}\int_a^x (x^{n-k} - t^{n-k})(-t)^k f(t)\,dt \\
&= \frac{1}{n!}\sum_{k=0}^{n-1} {}_{n}\mathrm{C}_k \int_a^x (x^{n-k} - t^{n-k})(-t)^k f(t)\,dt \\
&= \frac{1}{n!}\sum_{k=0}^{n} {}_{n}\mathrm{C}_k \int_a^x (x^{n-k} - t^{n-k})(-t)^k f(t)\,dt \quad (\because\ x^0 - t^0 = 0) \\
&= \frac{1}{n!}\int_a^x \left(\sum_{k=0}^{n} {}_{n}\mathrm{C}_k x^{n-k}(-t)^k - \sum_{k=0}^{n} {}_{n}\mathrm{C}_k t^{n-k}(-t)^k\right) f(t)\,dt \\
&= \frac{1}{n!}\int_a^x \left((x-t)^n - (t-t)^n\right) f(t)\,dt \\
&= \frac{1}{n!}\int_a^x (x-t)^n f(t)\,dt
\end{aligned}$$