公式（微分）

導関数の定義

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{t\to x}\frac{f(t)-f(x)}{t-x}$$

基本的関数の導関数

$$\Big(C\Big)' = 0$$

$$\Big(x^{\alpha}\Big)' = \alpha x^{\alpha-1} \qquad \left(\frac{1}{x^{\alpha}}\right)' = -\frac{\alpha}{x^{\alpha+1}}$$

$$\Big(e^{x}\Big)' = e^{x} \qquad \Big(a^{x}\Big)' = (\log a)a^{x}$$

$$\Big(\log x\Big)' = \frac{1}{x} \qquad \Big(\log_a x\Big)' = \frac{1}{(\log a)x}$$

$$\Big(\sin x\Big)' = \cos x \qquad \Big(\cos x\Big)' = -\sin x$$

$$\Big(\tan x\Big)' = \frac{1}{\cos^2 x} = 1+\tan^2 x \qquad \left(\frac{1}{\tan x}\right)' = -\frac{1}{\sin^2 x} = -1-\frac{1}{\tan^2 x}$$

$$\Big(\sin^{-1} x\Big)' = \frac{1}{\sqrt{1-x^2}} \qquad \Big(\cos^{-1} x\Big)' = -\frac{1}{\sqrt{1-x^2}}$$

$$\Big(\tan^{-1} x\Big)' = \frac{1}{1+x^2}$$

$$\Big(\sinh x\Big)' = \cosh x \qquad \Big(\cosh x\Big)' = \sinh x$$

$$\Big(\tanh x\Big)' = \frac{1}{\cosh^2 x} = 1-\tanh^2 x \qquad \left(\frac{1}{\tanh x}\right)' = -\frac{1}{\sinh^2 x} = 1-\frac{1}{\tanh^2 x}$$

$$\Big(\sinh^{-1} x\Big)' = \frac{1}{\sqrt{x^2+1}} \qquad \Big(\cosh^{-1} x\Big)' = \frac{1}{\sqrt{x^2-1}}$$

$$\Big(\tanh^{-1} x\Big)' = \frac{1}{1-x^2}$$

微分法の公式

定数倍 $\Big(a\cdot f(x)\Big)' = a\cdot f'(x)$ $\qquad \left(\frac{f(x)}{a}\right)' = \frac{f'(x)}{a}$

足し算 $\Big(f(x)+g(x)\Big)' = f'(x)+g'(x)$

引き算 $\Big(f(x)-g(x)\Big)' = f'(x)-g'(x)$

掛け算 $\Big(f(x)\cdot g(x)\Big)' = f'(x)\cdot g(x)+f(x)\cdot g'(x)$

割り算 $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{\big(g(x)\big)^2}$ $\qquad \left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{\big(g(x)\big)^2}$

合成関数の微分

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx} \qquad \Big(f\left(g(x)\right)\Big)'=f'\left(g(x)\right)\cdot g'(x)$$

特に

$$\Big(f(ax+b)\Big)'=a\cdot f'\left(ax+b\right)$$
$$\left(f\left(\frac{x+b}{a}\right)\right)'=\frac{1}{a}\cdot f'\left(\frac{x+b}{a}\right)$$
$$\Big(\big(g(x)\big)^a\Big)'=a\cdot g'(x)\cdot\big(g(x)\big)^{a-1}$$
$$\Big(\log|g(x)|\Big)'=\frac{g'(x)}{g(x)}$$
$$\Big(e^{g(x)}\Big)'=g'(x)\cdot e^{g(x)}$$

逆関数の微分

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}} \qquad \Big(f^{-1}(x)\Big)'=\frac{1}{f'(y)}=\frac{1}{f'\left(f^{-1}(x)\right)}$$

媒介変数表示関数の微分

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$x=f(t)$, $y=g(t)$ のとき

$$y'=\frac{g'(t)}{f'(t)}$$

対数微分法

$$y'=y\cdot\Big(\log|y|\Big)' \qquad \Big(f(x)\Big)'=f(x)\cdot\Big(\log|f(x)|\Big)'$$

増減，凹凸

区間 $a < x < b$ において

$f'(x)$	$+$	$+$	$-$	$-$
$f''(x)$	$+$	$-$	$+$	$-$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$f(x)$	╯	╭	╰	╮

極大・極小

$f(x)$ が $x = a$ で極大または極小 $\quad\Rightarrow\quad f'(a) = 0$ （必要条件）

$f'(a) = 0$ のとき

$f''(a) > 0 \quad\Rightarrow\quad x = a$ で極小 $\qquad ((a, f(a))$ が極小点)

$f''(a) < 0 \quad\Rightarrow\quad x = a$ で極大 $\qquad ((a, f(a))$ が極大点)

$f''(a) = 0 \quad\Rightarrow\quad$ これだけではわからない $\quad ((a, f(a))$ は極小点か極大点か踊り場)

接線

$y = f(x)$ の上の $x = x_0$である点における接線は

$$y = f'(x_0)(x - x_0) + f(x_0)$$

曲率円

$y = f(x)$ の上の $x = x_0$である点における曲率円の半径 r と中心 (a, b) は

$$\begin{aligned} r &= \frac{\left(1 + (y_0')^2\right)^{\frac{3}{2}}}{|y_0''|} \\ a &= x_0 - \frac{1 + (y_0')^2}{y_0''} \cdot y_0' \\ b &= y_0 + \frac{1 + (y')^2}{y_0''} \end{aligned} \qquad \begin{pmatrix} y_0 = f(x_0) \\ y_0' = f'(x_0) \\ y_0'' = f''(x_0) \end{pmatrix}$$

テーラー展開

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

右辺が有限の値に収束すれば，左辺の値はそれと等しい

マクローリン展開

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots \end{aligned}$$

右辺が有限の値に収束すれば，左辺の値はそれと等しい

$-\infty < x < +\infty$ で成り立つ例

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{k!}x^k + \cdots \\ \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^k \frac{1}{(2k+1)!}x^{2k+1} + \cdots \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^k \frac{1}{(2k)!}x^{2k} + \cdots \\ \sinh x &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots + \frac{1}{(2k+1)!}x^{2k+1} + \cdots \\ \cosh x &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots + \frac{1}{(2k)!}x^{2k} + \cdots \end{aligned}$$

$-1 < x < +1$ で成り立つ例

$$\begin{aligned} (1+x)^p &= 1 + \frac{p}{1}x + \frac{p}{1}\frac{p-1}{2}x^2 + \frac{p}{1}\frac{p-1}{2}\frac{p-2}{3}x^3 + \cdots \\ &\qquad + \left\{ \frac{p}{1}\frac{p-1}{2}\frac{p-2}{3} \cdots\cdots \frac{p+1-k}{k} \right\} x^k + \cdots \\ \frac{1}{\sqrt{1-x}} &= 1 + \frac{1}{2}x + \frac{1}{2}\frac{3}{4}x^2 + \frac{1}{2}\frac{3}{4}\frac{5}{6}x^3 + \cdots + \\ &\qquad + \left\{ \frac{1}{2}\frac{3}{4}\frac{5}{6} \cdots\cdots \frac{2k-1}{2k} \right\} x^k + \cdots \\ \sin^{-1} x &= x + \frac{1}{2}\frac{1}{3}x^3 + \frac{1}{2}\frac{3}{4}\frac{1}{5}x^5 + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{7}x^7 + \cdots + \\ &\qquad + \left\{ \frac{1}{2}\frac{3}{4}\frac{5}{6} \cdots\cdots \frac{2k-1}{2k}\frac{1}{2k+1} \right\} x^{2k+1} + \cdots \\ \tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + (-1)^k \frac{1}{2k+1}x^{2k+1} + \cdots \\ \log(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{k+1}\frac{1}{k}x^k + \cdots \end{aligned}$$

証明

$$\begin{aligned}(af(x))' &= \lim_{h\to 0}\frac{af(x+h)-af(x)}{h}\\ &= \lim_{h\to 0}\frac{a\left(f(x+h)-f(x)\right)}{h}\\ &= \lim_{h\to 0} a\cdot\frac{\left(f(x+h)-f(x)\right)}{h}\\ &= af'(x)\end{aligned}$$

または

$$\begin{aligned}(af(x))' &= \lim_{t\to x}\frac{af(t)-af(x)}{t-x}\\ &= \lim_{t\to x}\frac{a\left(f(t)-f(x)\right)}{t-x}\\ &= \lim_{t\to x} a\cdot\frac{\left(f(t)-f(x)\right)}{t-x}\\ &= af'(x)\end{aligned}$$

$$\begin{aligned}(f(x)+g(x))' &= \lim_{t\to x}\frac{(f(t)+g(t))-(f(x)+g(x))}{t-x}\\ &= \lim_{t\to x}\frac{f(t)+g(t)-f(x)-g(x)}{t-x}\\ &= \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}+\frac{g(t)-g(x}{t-x}\right)\\ &= f'(x)+g'(x)\end{aligned}$$

$$\begin{aligned}(f(x)-g(x))' &= \lim_{t\to x}\frac{(f(t)-g(t))-(f(x)-g(x))}{t-x}\\ &= \lim_{t\to x}\frac{f(t)-g(t)-f(x)+g(x)}{t-x}\\ &= \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}-\frac{g(t)-g(x}{t-x}\right)\\ &= f'(x)-g'(x)\end{aligned}$$

$$\begin{aligned}(f(x)g(x))' &= \lim_{t\to x}\frac{(f(t)g(t))-(f(x)g(x))}{t-x}\\ &= \lim_{t\to x}\frac{f(t)g(t)-f(x)g(t)+f(x)g(t)-f(x)g(x)}{t-x}\\ &= \lim_{t\to x}\frac{\left(f(t)-f(x)\right)g(t)+f(x)\left(g(t)-g(x)\right)}{t-x}\\ &= \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}g(t)+f(x)\frac{g(t)-g(x)}{t-x}\right)\\ &= f'(x)g(x)+f(x)g'(x)\end{aligned}$$

$$\begin{aligned}\left(\frac{f(x)}{g(x)}\right)' &= \lim_{t\to x}\frac{\dfrac{f(t)}{g(t)}-\dfrac{f(x)}{g(x)}}{t-x}\\ &= \lim_{t\to x}\frac{\dfrac{f(t)g(x)-f(x)g(t)}{g(t)g(x)}}{t-x}\\ &= \lim_{t\to x}\frac{f(t)g(x)-f(x)g(x)-f(x)g(t)+f(x)g(x)}{g(t)g(x)(t-x)}\\ &= \lim_{t\to x}\frac{\dfrac{\left(f(t)-f(x)\right)}{t-x}g(x)-f(x)\dfrac{\left(g(t)-g(x)\right)}{t-x}}{g(t)g(x)}\\ &= \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}\end{aligned}$$

$$
\begin{aligned}
(x^n)' &= \lim_{t\to x}\frac{t^n - x^n}{t - x}\\
&= \lim_{t\to x}\left(t^{n-1} + t^{n-2}x + \cdots + tx^{n-2} + x^{n-1}\right)\\
&= x^{n-1} + x^{n-1} + \cdots + x^{n-1} + x^{n-1}\\
&= nx^{n-1}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1}{x^n}\right)' &= \lim_{t\to x}\frac{\frac{1}{t^n} - \frac{1}{x^n}}{t - x}\\
&= \lim_{t\to x}\frac{x^n - t^n}{(t-x)t^n x^n}\\
&= \lim_{t\to x} -\frac{t^n - x^n}{(t-x)t^n x^n}\\
&= \lim_{t\to x} -\frac{t^{n-1} + t^{n-2}x + \cdots + tx^{n-2} + x^{n-1}}{t^n x^n}\\
&= -\frac{x^{n-1} + x^{n-1} + \cdots + x^{n-1} + x^{n-1}}{x^n x^n}\\
&= -\frac{nx^{n-1}}{x^{2n}}\\
&= -\frac{n}{x^{n+1}}
\end{aligned}
$$

$$
\begin{aligned}
(\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin h\cos x - (1-\cos h)\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin h\cos x - \dfrac{1-\cos^2 h}{1+\cos h}\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin h\cos x - \dfrac{\sin^2 h}{1+\cos h}\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin h}{h}\left(\cos x - \frac{\sin h}{1+\cos h}\sin x\right)\\
&= 1\left(\cos x - \frac{0}{2}\sin x\right)\\
&= \cos x
\end{aligned}
$$

$$
\begin{aligned}
(\cos x)' &= \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}\\
&= \lim_{h\to 0}\frac{-\sin h\sin x - (1-\cos h)\cos x}{h}\\
&= \lim_{h\to 0}\frac{-\sin h\sin x - \dfrac{1-\cos^2 h}{1+\cos h}\cos x}{h}\\
&= \lim_{h\to 0}\frac{-\sin h\sin x - \dfrac{\sin^2 h}{1+\cos h}\cos x}{h}\\
&= \lim_{h\to 0}\frac{\sin h}{h}\left(-\sin x - \frac{\sin h}{1+\cos h}\cos x\right)\\
&= 1\left(-\sin x - \frac{0}{2}\cos x\right)\\
&= -\sin x
\end{aligned}
$$

(注)　弧度法

$\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$が成り立つように角の単位を定めたのが弧度法である。

$$
\begin{aligned}
(\tan x)' &= \left(\frac{\sin x}{\cos x}\right)'\\
&= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{(\cos x)^2}\\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\
&= \begin{cases} \dfrac{1}{\cos^2 x} \\ 1 + \dfrac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1}{\tan x}\right)' &= \left(\frac{\cos x}{\sin x}\right)'\\
&= \frac{(\cos x)'\sin x - \cos x(\sin x)'}{(\sin x)^2}\\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}\\
&= \begin{cases} \dfrac{-1}{\sin^2 x} \\ -1 - \dfrac{\cos^2 x}{\sin^2 x} = -1 - \dfrac{1}{\tan^2 x} \end{cases}
\end{aligned}
$$

$$\left(e^x\right)' = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x\cdot e^h - e^x\cdot 1}{h} = \lim_{h\to 0}\frac{e^h-1}{h}e^x = e^x$$

$$\left(a^x\right)' = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x\cdot a^h - a^x\cdot 1}{h} = \lim_{h\to 0}\frac{a^h-1}{h}a^x = \lim_{h\to 0}\frac{e^{h\log a}-1}{h}a^x = \lim_{h\to 0}\frac{\left(e^{h\log a}-1\right)\log a}{h\log a}a^x = \lim_{H\to 0}\frac{e^H-1}{H}(\log a)\,a^x = (\log a)\,a^x$$

(注) e の定義

$\displaystyle\lim_{h\to 0}\frac{a^h-1}{h}=1$ となる a を e と名付けた。すなわち，$\displaystyle\lim_{h\to 0}\frac{e^h-1}{h}=1$

$$\left(\log x\right)' = \lim_{h\to 0}\frac{\log(x+h)-\log x}{h} = \lim_{h\to 0}\frac{\log\dfrac{x+h}{x}}{h} = \lim_{h\to 0}\frac{\log\left(1+\dfrac{h}{x}\right)}{\dfrac{h}{x}x} = \lim_{h\to 0}\frac{\log\left(1+\dfrac{h}{x}\right)}{\left(1+\dfrac{h}{x}\right)-1}\frac{1}{x} = \lim_{H\to 0}\frac{H}{e^H-1}\frac{1}{x} = \lim_{H\to 0}\frac{1}{\dfrac{e^H-1}{H}}\frac{1}{x} = \frac{1}{x}$$

または

$$y=\log x \iff x=e^y$$

$$\left(e^x\right)' = \frac{1}{(e^y)'} = \frac{1}{e^y} = \frac{1}{x}$$

$y=\sin^{-1}x \iff x=\sin y$ かつ $-\dfrac{\pi}{2}\leqq y\leqq\dfrac{\pi}{2}$

$$\left(\sin^{-1}x\right)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$y=\cos^{-1}x \iff x=\cos y$ かつ $0\leqq y\leqq\pi$

$$\left(\cos^{-1}x\right)' = \frac{1}{(\cos y)'} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}}$$

$$y=\tan^{-1}x \iff x=\tan y \quad \text{かつ} \quad -\frac{\pi}{2}<y<\frac{\pi}{2}$$

$$\begin{aligned}\left(\tan^{-1}x\right)' &= \frac{1}{(\tan y)'}\\ &= \frac{1}{1+\tan^2 y}\\ &= \frac{1}{1+x^2}\end{aligned}$$

$$\begin{aligned}\left(\sinh x\right)' &= \left(\frac{e^x-e^{-x}}{2}\right)'\\ &= \frac{e^x+e^{-x}}{2}\\ &= \cosh x\end{aligned}$$

$$\begin{aligned}\left(\cosh x\right)' &= \left(\frac{e^x+e^{-x}}{2}\right)'\\ &= \frac{e^x-e^{-x}}{2}\\ &= \sinh x\end{aligned}$$

$$\begin{aligned}\left(\tanh x\right)' &= \left(\frac{\sinh x}{\cosh x}\right)'\\ &= \frac{(\sinh x)'\cosh x-\sinh x\,(\cosh x)'}{(\cosh x)^2}\\ &= \frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}\\ &= \begin{cases}\dfrac{1}{\cosh^2 x}\\ 1-\tanh^2 x\end{cases}\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{\tanh x}\right)' &= \left(\frac{\cosh x}{\sinh x}\right)'\\ &= \frac{(\cosh x)'\cosh x-\sinh x\,(\sinh x)'}{(\sinh x)^2}\\ &= \frac{\sinh^2 x-\cosh^2 x}{\sinh^2 x}\\ &= \begin{cases}\dfrac{-1}{\sinh^2 x}\\ 1-\dfrac{1}{\tanh^2 x}\end{cases}\end{aligned}$$

$$y=\sinh^{-1}x \iff x=\sinh y$$

$$\begin{aligned}\left(\sinh^{-1}x\right)' &= \frac{1}{(\sinh y)'}\\ &= \frac{1}{\cosh y}\\ &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

$$y=\cosh^{-1}x \iff x=\cosh y \quad \text{かつ} \quad x \geqq 1$$

$$\begin{aligned}\left(\cosh^{-1}x\right)' &= \frac{1}{(\cosh y)'}\\ &= \frac{1}{\sinh y}\\ &= \frac{1}{\sqrt{x^2-1}}\end{aligned}$$

$$y=\tanh^{-1}x \iff x=\tanh y$$

$$\begin{aligned}\left(\tanh^{-1}x\right)' &= \frac{1}{(\tanh y)'}\\ &= \frac{1}{1-\tanh^2 y}\\ &= \frac{1}{1-x^2}\end{aligned}$$

(注)　逆双曲線関数の別表現

$$\sinh^{-1}x=\log\left(x+\sqrt{x^2+1}\right)$$

$$\cosh^{-1}x=\log\left(x+\sqrt{x^2-1}\right)$$

$$\tanh^{-1}x=\frac{1}{2}\log\frac{1+x}{1-x}$$