

## 公式 (積分)

不定積分の定義

$$\int f(x) dx = F(x) + C \quad \Leftrightarrow \quad F'(x) = f(x)$$

基本的関数の微分の逆

$$\int 0 dx = C$$

$$\int x^\alpha dx = \frac{1}{\alpha+1} x^{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\int e^x dx = e^x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \frac{1}{\cos^2 x} dx = \tan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$$

$$\int \sinh x dx = \cosh x + C$$

$$\int \frac{1}{\cosh^2 x} dx = \tanh x + C$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C$$

$$= \log |x + \sqrt{x^2-1}| + C$$

$$\int \frac{1}{1-x^2} dx = \tanh^{-1} x + C$$

$$= \frac{1}{2} \log \left| \frac{1+x}{1-x} \right| + C$$

$$\int \frac{1}{x} dx = \log |x| + C$$

$$\int a^x dx = \frac{a^x}{\log a} + C$$

$$\int \cos x dx = \sin x + C$$

$$\int (1 + \tan^2 x) dx = \tan x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int (1 - \tanh^2 x) dx = \tanh x + C$$

$$\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + C$$

$$= \log |x + \sqrt{x^2+1}| + C$$

積分法の公式

$$\int f(x) dx = F(x) + C, \quad \int g(x) dx = G(x) + C \quad \text{のとき}$$

$$\int (f(x) + g(x)) dx = F(x) + G(x) + C$$

$$\int a f(x) dx = a F(x) + C$$

$$\int f(ax+b) dx = \frac{1}{a} F(ax+b) + C$$

$$\int (f(x) - g(x)) dx = F(x) - G(x) + C$$

$$\int \frac{f(x)}{a} dx = \frac{F(x)}{a} + C$$

$$\int f\left(\frac{x+b}{a}\right) dx = a F\left(\frac{x+b}{a}\right) + C$$

## 定積分の定義

$$\int f(x) dx = F(x) + C \text{ のとき}$$

$$\int_a^b f(x) dx = \left[ F(x) \right]_a^b = F(b) - F(a)$$

## 部分積分法

$$\begin{aligned} \int u' \cdot v dx &= - \int u \cdot v' dx + u \cdot v & \int u \cdot v' dx &= - \int u' \cdot v dx + u \cdot v \\ \int_a^b u' \cdot v dx &= - \int_a^b u \cdot v' dx + \left[ u \cdot v \right]_a^b & \int_a^b u \cdot v' dx &= - \int_a^b u' \cdot v dx + \left[ u \cdot v \right]_a^b \end{aligned}$$

## 置換積分法 1

$t = g(x)$  とおくと

$$\begin{aligned} \int f(g(x)) g'(x) dx &= \int f(t) \frac{dt}{dx} dx = \int f(t) dt \\ &= F(t) + C = F(g(x)) + C \end{aligned}$$

$$\begin{aligned} \int_{x=a}^{x=b} f(g(x)) g'(x) dx &= \int_{x=a}^{x=b} f(t) \frac{dt}{dx} dx = \int_{t=g(a)}^{t=g(b)} f(t) dt \\ &= \left[ F(t) \right]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)) \end{aligned}$$

## 置換積分法 2

$x = h(t)$  とおくと

$$\int f(x) dx = \int f(h(t)) \frac{dx}{dt} dt = \int f(h(t)) h'(t) dt$$

$$\int_{x=a}^{x=b} f(x) dx = \int_{t=h^{-1}(a)}^{t=h^{-1}(b)} f(h(t)) h'(t) dt$$

## 置換の定石

次のように置換するとうまくいくことが多い。

$$\sqrt{ax+b} \quad \text{があるとき} \quad t = \sqrt{ax+b}$$

$$\text{すなわち} \quad x = \frac{t^2 - b}{a} \quad \text{とおく}$$

$$dx = \frac{2}{a} \cdot t dt$$

$$\sqrt{ax+b} = t \quad \text{となる}$$

$$\sqrt{x^2 + A} \quad \text{があるとき} \quad t = x + \sqrt{x^2 + A}$$

$$\text{すなわち} \quad x = \frac{1}{2} \left( t - \frac{A}{t} \right) \quad \text{とおく}$$

$$dx = \frac{1}{2} \left( 1 + \frac{A}{t^2} \right) dt$$

$$\sqrt{x^2 + A} = \frac{1}{2} \left( t + \frac{A}{t} \right) \quad \text{となる}$$

$$\sqrt{a^2 - x^2} \quad \text{があるとき} \quad \theta = \sin^{-1} \frac{x}{a}$$

$$\text{すなわち} \quad x = a \sin \theta \quad \text{とおく}$$

$$dx = a \cos \theta d\theta$$

$$\sqrt{a^2 - x^2} = a \cos \theta \quad \text{となる}$$

$$x^2 + a^2 \quad \text{があるとき} \quad \theta = \tan^{-1} \frac{x}{a}$$

$$\text{すなわち} \quad x = a \tan \theta \quad \text{とおく}$$

$$dx = \frac{a}{\cos^2 \theta} d\theta$$

$$x^2 + a^2 = \frac{a^2}{\cos^2 \theta} \quad \text{となる}$$

## 面積

$a < b$ ,  $f(x) \geq 0$  ( $a \leq x \leq b$ ) のとき,  $y = f(x)$  ( $a \leq x \leq b$ ) と  $x$  軸,  $x = a$ ,  $x = b$  で囲まれる部分の面積  $S$

$$S = \int_a^b f(x) dx$$

$a < b$  のとき,  $y = f(x)$ ,  $y = g(x)$  ( $a \leq x \leq b$ ) と  $x = a$ ,  $x = b$  で囲まれる部分の面積  $S$

$$S = \int_a^b |f(x) - g(x)| dx$$

## 長さ

$a < b$  のとき,  $y = f(x)$  ( $a \leq x \leq b$ ) の長さ  $\ell$

$$\ell = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

$\alpha < \beta$  のとき,  $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$  ( $\alpha \leq t \leq \beta$ ) の長さ  $\ell$

$$\ell = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_\alpha^\beta \sqrt{(f'(t))^2 + (g'(t))^2} dt$$

## 回転体の体積

$a < b$  のとき,  $y = f(x)$  ( $a \leq x \leq b$ ) と  $x$  軸,  $x = a$ ,  $x = b$  で囲まれる部分を  $x$  軸の周りに回転してできる

回転体の体積  $V_x$

$$V_x = \int_a^b \pi y^2 dx = \pi \int_a^b (f(x))^2 dx$$

$0 \leq a < b$ ,  $f(x) \geq 0$  ( $a \leq x \leq b$ ) のとき,  $y = f(x)$  ( $a \leq x \leq b$ ) と  $x$  軸,  $x = a$ ,  $x = b$  で囲まれる部分を  $y$  軸の周りに回転してできる回転体の体積  $V_y$

$$V_y = \int_a^b 2\pi xy dx = 2\pi \int_a^b xf(x) dx$$

## 回転面の面積

$a < b$  のとき,  $y = f(x)$  ( $a \leq x \leq b$ ) を  $x$  軸の周りに回転してできる回転面の面積  $S$

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$$

## 区分求積法

$$\int_a^b f(x) dx \quad \doteq \quad \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} f(x_k) \frac{b-a}{n}$$

$$\text{かつ} \quad \doteq \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \frac{b-a}{n}$$

$$\text{ただし} \quad x_k = a + \frac{(b-a)k}{n}$$

## 台形公式

$$\int_a^b f(x) dx \quad \doteq \quad \sum_{k=1}^n \frac{f(x_{k-1}) + f(x_k)}{2} \frac{b-a}{n} = \left( \frac{f(a) + f(b)}{2} + \sum_{k=1}^{n-1} f(x_k) \right) \frac{b-a}{n}$$

$$\text{ただし} \quad x_k = a + \frac{(b-a)k}{n}$$

## いろいろな積分(積分定数 +C は省略)

## 三角関数

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\begin{aligned} \int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\ &= \frac{x}{2} - \frac{\sin 2x}{4} \end{aligned}$$

$$\begin{aligned} \int \cos^2 x \, dx &= \int \frac{1 + \cos 2x}{2} \, dx \\ &= \frac{x}{2} + \frac{\sin 2x}{4} \end{aligned}$$

$$\begin{aligned} \int \sin^3 x \, dx &= \int \sin^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x)(-\cos x)' \, dx \\ &= -\cos x + \frac{\cos^3 x}{3} \end{aligned}$$

$$\begin{aligned} \int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx \\ &= \int (1 - \sin^2 x)(\sin x)' \, dx \\ &= \sin x - \frac{\sin^3 x}{3} \end{aligned}$$

$$\begin{aligned} \int \sin^3 x \, dx &= \int \frac{3 \sin x - \sin 3x}{4} \, dx \\ &= -\frac{3 \cos x}{4} + \frac{\cos 3x}{12} \end{aligned}$$

$$\begin{aligned} \int \cos^3 x \, dx &= \int \frac{3 \cos x + \cos 3x}{4} \, dx \\ &= \frac{3 \sin x}{4} + \frac{\sin 3x}{12} \end{aligned}$$

$$\begin{aligned} \int \sin^4 x \, dx &= \int \left( \frac{1 - \cos 2x}{2} \right)^2 \, dx \\ &= \int \frac{1 - 2 \cos 2x + \cos^2 2x}{4} \, dx \\ &= \int \frac{3 - 4 \cos 2x + \cos 4x}{8} \, dx \\ &= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \end{aligned}$$

$$\begin{aligned} \int \cos^4 x \, dx &= \int \left( \frac{1 + \cos 2x}{2} \right)^2 \, dx \\ &= \int \frac{1 + 2 \cos 2x + \cos^2 2x}{4} \, dx \\ &= \int \frac{3 + 4 \cos 2x + \cos 4x}{8} \, dx \\ &= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} \end{aligned}$$

$$\begin{aligned} \int \sin^5 x \, dx &= \int \sin^4 x \sin x \, dx \\ &= \int (1 - \cos^2 x)^2 (-\cos x)' \, dx \\ &= \int (1 - 2 \cos^2 x + \cos^4 x)(-\cos x)' \, dx \\ &= -\cos x + \frac{2 \cos^3 x}{3} - \frac{\cos^5 x}{5} \end{aligned}$$

$$\begin{aligned} \int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx \\ &= \int (1 - \sin^2 x)^2 (\sin x)' \, dx \\ &= \int (1 - 2 \sin^2 x + \sin^4 x)(\sin x)' \, dx \\ &= \sin x - \frac{2 \sin^3 x}{3} + \frac{\sin^5 x}{5} \end{aligned}$$

$$\begin{aligned}
\int \frac{1}{\sin x} dx &= \int \frac{\sin x}{\sin^2 x} dx \\
&= \int \frac{\sin x}{1 - \cos^2 x} dx \\
&= \frac{1}{2} \int \left( \frac{(-\cos x)'}{1 + \cos x} + \frac{(-\cos x)'}{1 - \cos x} \right) dx \\
&= \frac{1}{2} (-\log |1 + \cos x| + \log |1 - \cos x|) \\
&= \frac{1}{2} \log \left| \frac{1 - \cos x}{1 + \cos x} \right|
\end{aligned}
\qquad
\begin{aligned}
\int \frac{1}{\cos x} dx &= \int \frac{\cos x}{\cos^2 x} dx \\
&= \int \frac{\cos x}{1 - \sin^2 x} dx \\
&= \frac{1}{2} \int \left( \frac{(\sin x)'}{1 + \sin x} + \frac{(\sin x)'}{1 - \sin x} \right) dx \\
&= \frac{1}{2} (\log |1 + \sin x| - \log |1 - \sin x|) \\
&= \frac{1}{2} \log \left| \frac{1 + \sin x}{1 - \sin x} \right|
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{\sin^2 x} dx &= \int \frac{1}{\sin^2 x} dx \\
&= \int \frac{\cos^2 x}{\sin^2 x} \frac{1}{\cos^2 x} dx \\
&= \int \frac{1}{\tan^2 x} (\tan x)' dx \\
&= -\frac{1}{\tan x}
\end{aligned}
\qquad
\int \frac{1}{\cos^2 x} dx = \tan x$$

$$\begin{aligned}
\int \frac{1}{\sin^3 x} dx &= \int \frac{\sin x}{\sin^4 x} dx \\
&= \int \frac{\sin x}{(1 - \cos^2 x)^2} dx \\
&= \frac{1}{4} \int \left( \frac{(-\cos x)'}{1 + \cos x} + \frac{(-\cos x)'}{1 - \cos x} \right. \\
&\quad \left. + \frac{(-\cos x)'}{(1 + \cos x)^2} + \frac{(-\cos x)'}{(1 - \cos x)^2} \right) dx \\
&= \frac{1}{4} (-\log |1 + \cos x| + \log |1 - \cos x| \\
&\quad + \frac{1}{1 + \cos x} - \frac{1}{1 - \cos x}) \\
&= \frac{1}{4} \left( \log \left| \frac{1 - \cos x}{1 + \cos x} \right| + \frac{-2 \cos x}{1 - \cos^2 x} \right) \\
&= \frac{1}{4} \log \left| \frac{1 - \cos x}{1 + \cos x} \right| - \frac{1}{2} \frac{\cos x}{\sin^2 x}
\end{aligned}
\qquad
\begin{aligned}
\int \frac{1}{\cos^3 x} dx &= \int \frac{\cos x}{\cos^4 x} dx \\
&= \int \frac{\cos x}{(1 - \sin^2 x)^2} dx \\
&= \frac{1}{4} \int \left( \frac{(\sin x)'}{1 + \sin x} + \frac{(\sin x)'}{1 - \sin x} \right. \\
&\quad \left. + \frac{(\sin x)'}{(1 + \sin x)^2} + \frac{(\sin x)'}{(1 - \sin x)^2} \right) dx \\
&= \frac{1}{4} (\log |1 + \sin x| - \log |1 - \sin x| \\
&\quad - \frac{1}{1 + \sin x} + \frac{1}{1 - \sin x}) \\
&= \frac{1}{4} \left( \log \left| \frac{1 + \sin x}{1 - \sin x} \right| + \frac{2 \sin x}{1 - \sin^2 x} \right) \\
&= \frac{1}{4} \log \left| \frac{1 + \sin x}{1 - \sin x} \right| + \frac{1}{2} \frac{\sin x}{\cos^2 x}
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{\sin^4 x} dx &= \int \frac{(\sin^2 x + \cos^2 x) \cos^2 x}{\sin^4 x} \frac{1}{\cos^2 x} dx \\
&= \int \left( \frac{\cos^2 x}{\sin^2 x} + \frac{\cos^4 x}{\sin^4 x} \right) (\tan x)' dx \\
&= \int \left( \frac{1}{\tan^2 x} + \frac{1}{\tan^4 x} \right) (\tan x)' dx \\
&= -\frac{1}{\tan x} - \frac{1}{3 \tan^3 x}
\end{aligned}
\qquad
\begin{aligned}
\int \frac{1}{\cos^4 x} dx &= \int \frac{1}{\cos^2 x} \frac{1}{\cos^2 x} dx \\
&= \int (1 + \tan^2 x) (\tan x)' dx \\
&= \tan x + \frac{\tan^3 x}{3}
\end{aligned}$$

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= \int \frac{(-\cos x)'}{\cos x} \, dx \\ &= -\log |\cos x|\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\tan x} \, dx &= \int \frac{\cos x}{\sin x} \, dx \\ &= \int \frac{(\sin x)'}{\sin x} \, dx \\ &= \log |\sin x|\end{aligned}$$

$$\begin{aligned}\int \tan^2 x \, dx &= \int \frac{\sin^2 x}{\cos^2 x} \, dx \\ &= \int \frac{1 - \cos^2 x}{\cos^2 x} \, dx \\ &= \int \left( \frac{1}{\cos^2 x} - 1 \right) dx \\ &= \tan x - x\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\tan^2 x} \, dx &= \int \frac{\cos^2 x}{\sin^2 x} \, dx \\ &= \int \frac{1 - \sin^2 x}{\sin^2 x} \, dx \\ &= \int \left( \frac{1}{\sin^2 x} - 1 \right) dx \\ &= -\frac{1}{\tan x} - x\end{aligned}$$

$$\begin{aligned}\int \tan^3 x \, dx &= \int (\tan x(\tan^2 x + 1) - \tan x) \, dx \\ &= \int (\tan x(\tan x)' - \tan x) \, dx \\ &= \frac{\tan^2 x}{2} + \log |\cos x|\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\tan^3 x} \, dx &= \int \left( \frac{1 + \tan^2 x}{\tan^3 x} - \frac{1}{\tan x} \right) dx \\ &= \int \left( \frac{(\tan x)'}{\tan^3 x} - \frac{1}{\tan x} \right) dx \\ &= -\frac{1}{2 \tan^2 x} - \log |\sin x|\end{aligned}$$

## 逆三角関数

$$\begin{aligned}\int \sin^{-1} x \, dx &= \left( \text{部分積分} \begin{cases} u' = 1 \\ v = \sin^{-1} x \end{cases} \rightarrow \begin{cases} u = x \\ v' = \frac{1}{\sqrt{1-x^2}} \end{cases} \right) \\ &= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \sin^{-1} x + \sqrt{1-x^2}\end{aligned}$$

$$\begin{aligned}\int \tan^{-1} x \, dx &= \left( \text{部分積分} \begin{cases} u' = 1 \\ v = \tan^{-1} x \end{cases} \rightarrow \begin{cases} u = x \\ v' = \frac{1}{1+x^2} \end{cases} \right) \\ &= x \tan^{-1} x - \int \frac{x}{1+x^2} \, dx = x \tan^{-1} x - \frac{1}{2} \log |1+x^2|\end{aligned}$$



## 分数式

$$\int \frac{1}{ax+b} dx = \frac{\log|ax+b|}{a} \quad (a \neq 0)$$

$$\int \frac{1}{(ax+b)^n} dx = -\frac{1}{a(n-1)(ax+b)^{n-1}} \quad (a \neq 0, n \geq 2)$$

$$\int \frac{1}{(x-\alpha)^2} dx = -\frac{1}{x-\alpha}$$

$$\begin{aligned} \int \frac{1}{(x-\alpha)(x-\beta)} dx &= \frac{1}{\alpha-\beta} \int \left( \frac{1}{x-\alpha} - \frac{1}{x-\beta} \right) dx = \frac{1}{\alpha-\beta} (\log|x-\alpha| - \log|x-\beta|) \\ &= \frac{1}{\alpha-\beta} \log \left| \frac{x-\alpha}{x-\beta} \right| \quad (\alpha \neq \beta) \end{aligned}$$

$$\int \frac{1}{x^2-a^2} dx = \int \frac{1}{(x-a)(x+a)} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \quad (a \neq 0)$$

$$\begin{aligned} \int \frac{1}{x^2+a^2} dx &\quad (\text{置換積分 } x = a \tan \theta) \\ &= \int \frac{1}{a^2(\tan^2 \theta + 1)} a(1 + \tan^2 \theta) d\theta = \int \frac{1}{a} d\theta = \frac{\theta}{a} = \frac{1}{a} \tan^{-1} \frac{x}{a} \quad (a \neq 0) \end{aligned}$$

$$\int \frac{1}{(x-p)^2+q^2} dx = \frac{1}{q} \tan^{-1} \frac{x-p}{q} \quad (q \neq 0)$$

$$\int \frac{2ax+b}{ax^2+bx+c} dx = \int \frac{(ax^2+bx+c)'}{ax^2+bx+c} dx = \log|ax^2+bx+c|$$

$$\begin{aligned} \int \frac{dx+e}{ax^2+bx+c} dx &= \int \frac{D(2ax+b)+E}{ax^2+bx+c} dx \\ &= D \log|ax^2+bx+c| + \begin{cases} \int \frac{E}{a(x-\alpha)(x-\beta)} dx & (b^2-4ac > 0 \text{ のとき}) \\ \int \frac{E}{a(x-\alpha)^2} dx & (b^2-4ac = 0 \text{ のとき}) \\ \int \frac{E}{a((x-p)^2+q^2)} dx & (b^2-4ac < 0 \text{ のとき}) \end{cases} \end{aligned}$$

と変形する

## 根号を含む式

$$\begin{aligned} \int \frac{1}{\sqrt{a^2 - x^2}} dx & \quad (\text{置換積分 } x = a \sin \theta) \\ &= \int \frac{1}{a \cos \theta} a \cos \theta d\theta \\ &= \int 1 d\theta = \theta \\ &= \sin^{-1} \frac{x}{a} \quad (a > 0) \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + A}} dx & \quad \left( \text{置換積分 } x = \frac{1}{2} \left( t - \frac{A}{t} \right) \right) \\ &= \int \frac{1}{\frac{1}{2} \left( t + \frac{A}{t} \right)} \frac{1}{2} \left( 1 + \frac{A}{t^2} \right) dt \\ &= \int \frac{1}{t} dt = \log |t| \\ &= \log \left| \left( x + \sqrt{x^2 + A} \right) \right| \end{aligned}$$

$$\begin{aligned} \int \frac{x}{\sqrt{a^2 - x^2}} dx &= \int \frac{-\frac{1}{2}(a^2 - x^2)'}{\sqrt{a^2 - x^2}} dx \\ &= -\sqrt{a^2 - x^2} \end{aligned}$$

$$\begin{aligned} \int \frac{x}{\sqrt{x^2 + A}} dx &= \int \frac{\frac{1}{2}(x^2 + A)'}{\sqrt{x^2 + A}} dx \\ &= \sqrt{x^2 + A} \end{aligned}$$

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx & \quad (\text{置換積分 } x = a \sin \theta) \\ &= \int a \cos \theta a \cos \theta d\theta \\ &= a^2 \int \cos^2 \theta d\theta \\ &= a^2 \left( \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \\ &= a^2 \frac{\theta}{2} + \frac{a \sin \theta a \cos \theta}{2} \\ &= \frac{1}{2} \left( a^2 \sin^{-1} \frac{x}{a} + x \sqrt{a^2 - x^2} \right) \end{aligned}$$

$$\begin{aligned} \int \sqrt{x^2 + A} dx & \quad \left( \text{置換積分 } x = \frac{1}{2} \left( t - \frac{A}{t} \right) \right) \\ &= \int \frac{1}{2} \left( t + \frac{A}{t} \right) \frac{1}{2} \left( 1 + \frac{A}{t^2} \right) dt \\ &= \frac{1}{4} \int \left( t + \frac{2A}{t} + \frac{A^2}{t^3} \right) dx \\ &= \frac{1}{4} \left( \frac{t^2}{2} + 2A \log |t| - \frac{A^2}{2t^2} \right) \\ &= \frac{1}{2} \left( \frac{1}{2} \left( t - \frac{A}{t} \right) \frac{1}{2} \left( t + \frac{A}{t} \right) + A \log |t| \right) \\ &= \frac{1}{2} \left( x \sqrt{x^2 + A} + A \log \left| x + \sqrt{x^2 + A} \right| \right) \end{aligned}$$

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &= \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx \\ &= \int x \cdot \frac{-x}{\sqrt{a^2 - x^2}} dx \\ & \quad + \int \frac{a^2}{\sqrt{a^2 - x^2}} dx \\ &= x \sqrt{a^2 - x^2} - \int 1 \sqrt{a^2 - x^2} dx \\ & \quad + a^2 \sin^{-1} \frac{x}{a} \end{aligned}$$

$$\begin{aligned} \int \sqrt{x^2 + A} dx &= \int \frac{x^2 + A}{\sqrt{x^2 + A}} dx \\ &= \int x \cdot \frac{x}{\sqrt{x^2 + A}} dx \\ & \quad + \int \frac{A}{\sqrt{x^2 + A}} dx \\ &= x \sqrt{x^2 + A} - \int 1 \sqrt{x^2 + A} dx \\ & \quad + A \log \left| x + \sqrt{x^2 + A} \right| \end{aligned}$$

ゆえに

$$\begin{aligned} 2 \int \sqrt{a^2 - x^2} dx &= x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \\ \int \sqrt{a^2 - x^2} dx &= \frac{1}{2} \left( x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right) \end{aligned}$$

ゆえに

$$\begin{aligned} 2 \int \sqrt{x^2 + A} dx &= x \sqrt{x^2 + A} + A \log \left| x + \sqrt{x^2 + A} \right| \\ \int \sqrt{x^2 + A} dx &= \frac{1}{2} \left( x \sqrt{x^2 + A} + A \log \left| x + \sqrt{x^2 + A} \right| \right) \end{aligned}$$

## 指数・対数関数

$$\int a^x dx = \int e^{x \log a} dx = \frac{e^{x \log a}}{\log a} = \frac{a^x}{\log a}$$

$$\int x e^x dx \quad \left( \text{部分積分} \begin{cases} u = x \\ v' = e^x \end{cases} \rightarrow \begin{cases} u' = 1 \\ v = e^x \end{cases} \right)$$

$$= x e^x - \int e^x dx = x e^x - e^x = (x-1)e^x$$

$$\int x e^{x^2} dx = \int \frac{1}{2} e^{x^2} (x^2)' dx = \frac{1}{2} e^{x^2}$$

$$\int \log x dx \quad \left( \text{部分積分} \begin{cases} u' = 1 \\ v = \log x \end{cases} \rightarrow \begin{cases} u = x \\ v' = \frac{1}{x} \end{cases} \right)$$

$$= x \log x - \int 1 dx = x \log x - x$$

$$\int (\log x)^2 dx \quad \left( \text{部分積分} \begin{cases} u' = 1 \\ v = (\log x)^2 \end{cases} \rightarrow \begin{cases} u = x \\ v' = \frac{2 \log x}{x} \end{cases} \right)$$

$$= x (\log x)^2 - \int 2 \log x dx = x (\log x)^2 - 2x \log x + 2x$$

$$\int x \log x dx \quad \left( \text{部分積分} \begin{cases} u' = x \\ v = \log x \end{cases} \rightarrow \begin{cases} u = \frac{x^2}{2} \\ v' = \frac{1}{x} \end{cases} \right)$$

$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx = \frac{x^2 \log x}{2} - \frac{x^2}{4}$$

$$\int \frac{\log x}{x} dx = \int \log x (\log x)' dx = \frac{(\log x)^2}{2}$$

$$\int \log(x + \sqrt{x^2 + A}) dx \quad \left( \text{部分積分} \begin{cases} u' = 1 \\ v = \log(x + \sqrt{x^2 + A}) \end{cases} \rightarrow \begin{cases} u = x \\ v' = \frac{1}{\sqrt{x^2 + A}} \end{cases} \right)$$

$$= x \log(x + \sqrt{x^2 + A}) - \int \frac{x}{\sqrt{x^2 + A}} dx = x \log(x + \sqrt{x^2 + A}) - \sqrt{x^2 + A}$$

## 双曲線関数

$$\int \cosh x \, dx = \sinh x$$

$$\int \sinh x \, dx = \cosh x$$

$$\int \frac{e^x + e^{-x}}{2} \, dx = \frac{e^x - e^{-x}}{2}$$

$$\int \frac{e^x - e^{-x}}{2} \, dx = \frac{e^x + e^{-x}}{2}$$

$$\begin{aligned} \int \cosh^2 x \, dx &= \int \frac{\cosh 2x + 1}{2} \, dx \\ &= \frac{\sinh 2x}{4} + \frac{x}{2} \end{aligned}$$

$$\begin{aligned} \int \sinh^2 x \, dx &= \int \frac{\cosh 2x - 1}{2} \, dx \\ &= \frac{\sinh 2x}{4} - \frac{x}{2} \end{aligned}$$

$$\begin{aligned} \int \left( \frac{e^x + e^{-x}}{2} \right)^2 \, dx &= \int \frac{e^{2x} + e^{-2x} + 2}{4} \, dx \\ &= \frac{e^{2x} - e^{-2x}}{8} + \frac{x}{2} \end{aligned}$$

$$\begin{aligned} \int \left( \frac{e^x - e^{-x}}{2} \right)^2 \, dx &= \int \frac{e^{2x} + e^{-2x} - 2}{4} \, dx \\ &= \frac{e^{2x} - e^{-2x}}{8} - \frac{x}{2} \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\cosh x} \, dx &= \int \frac{\cosh x}{\cosh^2 x} \, dx \\ &= \int \frac{(\sinh x)'}{\sinh^2 x + 1} \, dx \\ &= \tan^{-1} \sinh x \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sinh x} \, dx &= \int \frac{\sinh x}{\sinh^2 x} \, dx \\ &= \int \frac{(\cosh x)'}{\cosh^2 x - 1} \, dx \\ &= \frac{1}{2} \log \left| \frac{\cosh x - 1}{\cosh x + 1} \right| \end{aligned}$$

$$\begin{aligned} \int \frac{2}{e^x + e^{-x}} \, dx &= \int \frac{2e^x}{(e^x)^2 + 1} \, dx \\ &= 2 \tan^{-1} e^x \end{aligned}$$

$$\begin{aligned} \int \frac{2}{e^x - e^{-x}} \, dx &= \int \frac{2e^x}{(e^x)^2 - 1} \, dx \\ &= \log \left| \frac{e^x - 1}{e^x + 1} \right| \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\cosh^2 x} \, dx &= \int \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \, dx \\ &= \frac{\sinh x}{\cosh x} \\ &= \tanh x \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sinh^2 x} \, dx &= \int \frac{-(\sinh^2 x - \cosh^2 x)}{\sinh^2 x} \, dx \\ &= \frac{-\cosh x}{\sinh x} \\ &= -\frac{1}{\tanh x} \end{aligned}$$

$$\begin{aligned} \int \left( \frac{2}{e^x + e^{-x}} \right)^2 \, dx &= \int \frac{2(2e^{2x})}{(e^{2x} + 1)^2} \, dx \\ &= -\frac{2}{e^{2x} + 1} \end{aligned}$$

$$\begin{aligned} \int \left( \frac{2}{e^x - e^{-x}} \right)^2 \, dx &= \int \frac{2(2e^{2x})}{(e^{2x} - 1)^2} \, dx \\ &= -\frac{2}{e^{2x} - 1} \end{aligned}$$

$$\begin{aligned}\int \tanh x \, dx &= \int \frac{\sinh x}{\cosh x} \, dx \\ &= \log |\cosh x|\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\tanh x} \, dx &= \int \frac{\cosh x}{\sinh x} \, dx \\ &= \log |\sinh x|\end{aligned}$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx = \log |e^x + e^{-x}|$$

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx = \log |e^x - e^{-x}|$$

$$\begin{aligned}\int \tanh^2 x \, dx &= \int (-(1 - \tanh^2 x) + 1) \, dx \\ &= -\tanh x + x\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\tanh^2 x} \, dx &= \int \left( \frac{1 - \tanh^2 x}{\tanh^2 x} + 1 \right) \, dx \\ &= -\frac{1}{\tanh x} + x\end{aligned}$$

$$\begin{aligned}\int \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2 \, dx &= \int \left( \frac{e^{2x} - 1}{e^{2x} + 1} \right)^2 \, dx \\ &= \int \frac{(e^{2x} + 1)^2 - 4e^{2x}}{(e^{2x} + 1)^2} \, dx \\ &= \int \left( 1 - \frac{2(2e^{2x})}{(e^{2x} + 1)^2} \right) \, dx \\ &= x + \frac{2}{e^{2x} + 1}\end{aligned}$$

$$\begin{aligned}\int \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right)^2 \, dx &= \int \left( \frac{e^{2x} + 1}{e^{2x} - 1} \right)^2 \, dx \\ &= \int \frac{(e^{2x} - 1)^2 + 4e^{2x}}{(e^{2x} - 1)^2} \, dx \\ &= \int \left( 1 + \frac{2(2e^{2x})}{(e^{2x} - 1)^2} \right) \, dx \\ &= x - \frac{2}{e^{2x} - 1}\end{aligned}$$

## その他の不定積分

$$\begin{aligned}C(x) &= \int e^{ax} \cos bx \, dx \\&= \frac{e^{ax}}{a} \cos bx - \int \frac{e^{ax}}{a} (-b \sin bx) \, dx \\&= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} S(x) \\&= \frac{e^{ax}}{a} \cos bx + \frac{b}{a} \left( \frac{e^{ax}}{a} \sin bx - \frac{b}{a} C(x) \right) \\a^2 C(x) &= e^{ax} (a \cos bx + b \sin bx) - b^2 C(x)\end{aligned}$$

$$\begin{aligned}(a^2 + b^2)C(x) &= e^{ax} (a \cos bx + b \sin bx) \\C(x) &= \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)\end{aligned}$$

$$\begin{aligned}S(x) &= \int e^{ax} \sin bx \, dx \\&= \frac{e^{ax}}{a} \sin bx - \int \frac{e^{ax}}{a} (b \cos bx) \, dx \\&= \frac{e^{ax}}{a} \sin bx - \frac{b}{a} C(x) \\&= \frac{e^{ax}}{a} \sin bx - \frac{b}{a} \left( \frac{e^{ax}}{a} \cos bx + \frac{b}{a} S(x) \right) \\a^2 S(x) &= e^{ax} (a \sin bx - b \cos bx) - b^2 S(x)\end{aligned}$$

$$\begin{aligned}(a^2 + b^2)S(x) &= e^{ax} (a \sin bx - b \cos bx) \\S(x) &= \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)\end{aligned}$$

## 定積分

$$\begin{aligned}
\int_{\alpha}^{\beta} (x - \alpha)(x - \beta) dx &= \left[ \frac{(x - \alpha)^2}{2} (x - \beta) \right]_{\alpha}^{\beta} - a \int_{\alpha}^{\beta} \frac{(x - \alpha)^2}{2} \cdot 1 dx \\
&= - \left[ \frac{(x - \alpha)^3}{6} \right]_{\alpha}^{\beta} \\
&= - \frac{(\beta - \alpha)^3}{6}
\end{aligned}$$

$$\begin{aligned}
\int_{\alpha}^{\beta} (x - \alpha)(x - \beta) dx &= \int_{\alpha}^{\beta} (x - \alpha)((x - \alpha) - (\beta - \alpha)) dx \\
&= \int_{\alpha}^{\beta} ((x - \alpha)^2 - (\beta - \alpha)(x - \alpha)) dx \\
&= \left[ \frac{(x - \alpha)^3}{3} - (\beta - \alpha) \frac{(x - \alpha)^2}{2} \right]_{\alpha}^{\beta} \\
&= \left( \frac{(\beta - \alpha)^3}{3} - \frac{(\beta - \alpha)^3}{2} \right) \\
&= - \frac{(\beta - \alpha)^3}{6}
\end{aligned}$$

$$\begin{aligned}
\int_{\alpha}^{\gamma} (x - \alpha)(x - \beta)(x - \gamma) dx &= \int_{\alpha}^{\gamma} (x - \alpha)((x - \alpha) - (\beta - \alpha))((x - \alpha) - (\gamma - \alpha)) dx \\
&= \int_{\alpha}^{\gamma} ((x - \alpha)^3 - ((\beta - \alpha) + (\gamma - \alpha))(x - \alpha)^2 + (\beta - \alpha)(\gamma - \alpha)(x - \alpha)) dx \\
&= \left[ \frac{(x - \alpha)^4}{4} - ((\beta - \alpha) + (\gamma - \alpha)) \frac{(x - \alpha)^3}{3} + (\beta - \alpha)(\gamma - \alpha) \frac{(x - \alpha)^2}{2} \right]_{\alpha}^{\gamma} \\
&= \frac{(\gamma - \alpha)^3}{12} (3(\gamma - \alpha) - 4(\beta - \alpha) - 4(\gamma - \alpha) + 6(\beta - \alpha)) \\
&= - \frac{(\gamma - \alpha)^3(\gamma - 2\beta + \alpha)}{12}
\end{aligned}$$

$$\begin{aligned}
\int_{\alpha}^{\beta} (x - \alpha)^n (x - \beta) dx &= \left[ \frac{(x - \alpha)^{n+1}}{n+1} (x - \beta) \right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \frac{(x - \alpha)^{n+1}}{n+1} \cdot 1 dx \\
&= - \left[ \frac{(x - \alpha)^{n+2}}{(n+2)(n+1)} \right]_{\alpha}^{\beta} \\
&= - \frac{(\beta - \alpha)^{n+2}}{(n+2)(n+1)}
\end{aligned}$$

$$\begin{aligned}
S_{m,n} &= \int_{\alpha}^{\beta} (x-\alpha)^m (x-\beta)^n dx \\
&= \left[ \frac{(x-\alpha)^{m+1}}{m+1} (x-\beta)^n \right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \frac{(x-\alpha)^{m+1}}{m+1} \cdot n(x-\beta)^{n-1} dx \\
&= -\frac{n}{m+1} S_{m+1,n-1} \\
&= +\frac{n}{m+1} \frac{n-1}{m+2} S_{m+2,n-2} \\
&= \dots \\
&= (-1)^n \frac{n(n-1)\dots 1}{(m+1)(m+2)\dots(m+n)} S_{m+n,0} \\
&= (-1)^n \frac{m!n!}{(m+n)!} \int_{\alpha}^{\beta} (x-\alpha)^{m+n} dx \\
&= (-1)^n \frac{m!n!}{(m+n+1)!} (\beta-\alpha)^{m+n+1}
\end{aligned}$$

$$\begin{aligned}
S_n &= \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin x \sin^{n-1} x dx \\
&= \left[ -\cos x \sin^{n-1} x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x (n-1) \sin^{n-2} x \cos x dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x) dx \\
&= \int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x dx \\
&= (n-1)S_{n-2} - (n-1)S_n
\end{aligned}$$

ゆえに

$$\begin{aligned}
S_n &= \frac{n-1}{n} S_{n-2} = \frac{n-1}{n} \frac{n-3}{n-2} S_{n-4} = \dots \\
&= \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{1}{2} S_0 & (n: \text{偶数}) \\ \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{2}{3} S_1 & (n: \text{奇数}) \end{cases} \\
&= \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{1}{2} \cdot \frac{\pi}{2} & (n: \text{偶数}) \\ \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{2}{3} \cdot 1 & (n: \text{奇数}) \end{cases}
\end{aligned}$$

同様に

$$\begin{aligned}
C_n &= \int_0^{\frac{\pi}{2}} \cos^n x dx \\
&= \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{1}{2} \cdot \frac{\pi}{2} & (n: \text{偶数}) \\ \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{2}{3} \cdot 1 & (n: \text{奇数}) \end{cases}
\end{aligned}$$