公式（積分）

不定積分の定義

$$\int f(x)\,dx = F(x) + C \qquad \Longleftrightarrow \qquad F'(x) = f(x)$$

基本的関数の微分の逆

$$\int 0\,dx = C$$

$$\int x^\alpha\,dx = \frac{1}{\alpha+1}\,x^{\alpha+1} + C \quad (\alpha \fallingdotseq -1)$$

$$\int \frac{1}{x}\,dx = \log|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\log a} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + C$$

$$\int (1+\tan^2 x)\,dx = \tan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \frac{1}{\cosh^2 x}\,dx = \tanh x + C$$

$$\int (1-\tanh^2 x)\,dx = \tanh x + C$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C = \log\left|x + \sqrt{x^2-1}\right| + C$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C = \log\left|x + \sqrt{x^2+1}\right| + C$$

$$\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + C = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C$$

積分法の公式

$\displaystyle\int f(x)\,dx = F(x) + C$, $\displaystyle\int g(x)\,dx = G(x) + C$ のとき

$$\int (f(x)+g(x))\,dx = F(x) + G(x) + C$$

$$\int (f(x)-g(x))\,dx = F(x) - G(x) + C$$

$$\int af(x)\,dx = a\,F(x) + C$$

$$\int \frac{f(x)}{a}\,dx = \frac{F(x)}{a} + C$$

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$$

$$\int f\left(\frac{x+b}{a}\right)dx = a\,F\left(\frac{x+b}{a}\right) + C$$

定積分の定義

$\int f(x)\,dx = F(x) + C$ のとき

$$\int_a^b f(x)\,dx = \Big[\,F(x)\,\Big]_a^b = F(b) - F(a)$$

部分積分法

$$\int u' \cdot v\,dx = -\int u \cdot v'\,dx + u \cdot v \qquad \int u \cdot v'\,dx = -\int u' \cdot v\,dx + u \cdot v$$

$$\int_a^b u' \cdot v\,dx = -\int_a^b u \cdot v'\,dx + \Big[\,u \cdot v\,\Big]_a^b \qquad \int_a^b u \cdot v'\,dx = -\int_a^b u' \cdot v\,dx + \Big[\,u \cdot v\,\Big]_a^b$$

置換積分法 1

$t = g(x)$ とおくと

$$\begin{aligned}\int f\left(g(x)\right)\ g'(x)\,dx &= \int f(t)\ \frac{dt}{dx}\,dx &&= \int f(t)\,dt \\ &= F(t) + C &&= F(g(x)) + C\end{aligned}$$

$$\begin{aligned}\int_{x=a}^{x=b} f\left(g(x)\right)\ g'(x)\,dx &= \int_{x=a}^{x=b} f(t)\ \frac{dt}{dx}\,dx &&= \int_{t=g(a)}^{t=g(b)} f(t)\,dt \\ &= \Big[\,F(t)\,\Big]_{g(a)}^{g(b)} &&= F\left(g(b)\right) - F((g(a))\end{aligned}$$

置換積分法 2

$x = h(t)$ とおくと

$$\int f(x)\,dx = \int f\left(h(t)\right)\ \frac{dx}{dt}\,dt = \int f\left(h(t)\right)\ h'(t)\,dt$$

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{t=h^{-1}(a)}^{t=h^{-1}(b)} f(h(t))h'(t)\,dt$$

置換の定石

次のように置換するとうまくいくことが多い。

$\sqrt{ax+b}$ があるとき $t=\sqrt{ax+b}$

すなわち $x=\dfrac{t^2-b}{a}$ とおく

$dx=\dfrac{2}{a}\cdot t\,dt$

$\sqrt{ax+b}=t$ となる

$\sqrt{x^2+A}$ があるとき $t=x+\sqrt{x^2+A}$

すなわち $x=\dfrac{1}{2}\left(t-\dfrac{A}{t}\right)$ とおく

$dx=\dfrac{1}{2}\left(1+\dfrac{A}{t^2}\right)dt$

$\sqrt{x^2+A}=\dfrac{1}{2}\left(t+\dfrac{A}{t}\right)$ となる

$\sqrt{a^2-x^2}$ があるとき $\theta=\sin^{-1}\dfrac{x}{a}$

すなわち $x=a\,\sin\theta$ とおく

$dx=a\,\cos\theta\,d\theta$

$\sqrt{a^2-x^2}=a\cos\theta$ となる

x^2+a^2 があるとき $\theta=\tan^{-1}\dfrac{x}{a}$

すなわち $x=a\tan\theta$ とおく

$dx=\dfrac{a}{\cos^2\theta}\,d\theta$

$x^2+a^2=\dfrac{a^2}{\cos^2\theta}$ となる

面積

$a < b,\ f(x) \geqq 0\ (a \leqq x \leqq b)$ のとき，$y = f(x)\ (a \leqq x \leqq b)$ と x 軸, $x = a, x = b$ で囲まれる部分の面積 S

$$S = \int_a^b f(x)\,dx$$

$a < b$ のとき，$y = f(x), y = g(x)\ (a \leqq x \leqq b)$ と $x = a, x = b$ で囲まれる部分の面積 S

$$S = \int_a^b |f(x) - g(x)|\,dx$$

長さ

$a < b$ のとき，$y = f(x)\ (a \leqq x \leqq b)$ の長さ ℓ

$$\ell = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

$\alpha < \beta$ のとき，$\begin{cases} x = f(t) \\ y = g(t) \end{cases}\ (\alpha \leqq t \leqq \beta)$ の長さ ℓ

$$\ell = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_\alpha^\beta \sqrt{(f'(t))^2 + (g'(t))^2}\,dt$$

回転体の体積

$a < b$ のとき，$y = f(x)\ (a \leqq x \leqq b)$ と x 軸, $x = a, x = b$ で囲まれる部分を x 軸の周りに回転してできる回転体の体積 V_x

$$V_x = \int_a^b \pi y^2\,dx = \pi \int_a^b (f(x))^2\,dx$$

$0 \leqq a < b,\ f(x) \geqq 0\ (a \leqq x \leqq b)$ のとき，$y = f(x)\ (a \leqq x \leqq b)$ と x 軸, $x = a, x = b$ で囲まれる部分を y 軸の周りに回転してできる回転体の体積 V_y

$$V_y = \int_a^b 2\pi xy\,dx = 2\pi \int_a^b xf(x)\,dx$$

回転面の面積

$a < b$ のとき，$y = f(x)\ (a \leqq x \leqq b)$ を x 軸の周りに回転してできる回転面の面積 S

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$$

区分求積法

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{k=0}^{n-1}f(x_k) = \int_a^b f(x)\,dx$$

$$\lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n}f(x_k) = \int_a^b f(x)\,dx$$

ただし $x_k = a + \dfrac{(b-a)k}{n}$

特に

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx$$

台形公式

$$\int_a^b f(x)\,dx \fallingdotseq \frac{b-a}{n}\sum_{k=1}^{n}\frac{f(x_{k-1})+f(x_k)}{2} = \frac{b-a}{n}\left(\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n-1}f(x_k)\right)$$

ただし $x_k = a + \dfrac{(b-a)k}{n}$

いろいろな積分（積分定数 $+C$ は省略）

三角関数

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

$$\begin{aligned}\int \sin^2 x \, dx &= \int \frac{1-\cos 2x}{2} \, dx \\ &= \frac{x}{2} - \frac{\sin 2x}{4}\end{aligned}$$

$$\begin{aligned}\int \cos^2 x \, dx &= \int \frac{1+\cos 2x}{2} \, dx \\ &= \frac{x}{2} + \frac{\sin 2x}{4}\end{aligned}$$

$$\begin{aligned}\int \sin^3 x \, dx &= \int \sin^2 x \sin x \, dx \\ &= \int (1-\cos^2 x)(-\cos x)' \, dx \\ &= -\cos x + \frac{\cos^3 x}{3}\end{aligned}$$

$$\begin{aligned}\int \sin^3 x \, dx &= \int \frac{3\sin x - \sin 3x}{4} \, dx \\ &= -\frac{3\cos x}{4} + \frac{\cos 3x}{12}\end{aligned}$$

$$\begin{aligned}\int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx \\ &= \int (1-\sin^2 x)(\sin x)' \, dx \\ &= \sin x - \frac{\sin^3 x}{3}\end{aligned}$$

$$\begin{aligned}\int \cos^3 x \, dx &= \int \frac{3\cos x + \cos 3x}{4} \, dx \\ &= \frac{3\sin x}{4} + \frac{\sin 3x}{12}\end{aligned}$$

$$\begin{aligned}\int \sin^4 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx \\ &= \int \frac{1-2\cos 2x+\cos^2 2x}{4} \, dx \\ &= \int \frac{3-4\cos 2x+\cos 4x}{8} \, dx \\ &= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\end{aligned}$$

$$\begin{aligned}\int \cos^4 x \, dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \int \frac{1+2\cos 2x+\cos^2 2x}{4} \, dx \\ &= \int \frac{3+4\cos 2x+\cos 4x}{8} \, dx \\ &= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\end{aligned}$$

$$\begin{aligned}\int \sin^5 x \, dx &= \int \sin^4 x \sin x \, dx \\ &= \int (1-\cos^2 x)^2(-\cos x)' \, dx \\ &= \int (1-2\cos^2 x+\cos^4 x)(-\cos x)' \, dx \\ &= -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5}\end{aligned}$$

$$\begin{aligned}\int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx \\ &= \int (1-\sin^2 x)^2(\sin x)' \, dx \\ &= \int (1-2\sin^2 x+\sin^4 x)(\sin x)' \, dx \\ &= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5}\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sin x}\,dx &= \int \frac{\sin x}{\sin^2 x}\,dx\\ &= \int \frac{\sin x}{1-\cos^2 x}\,dx\\ &= \frac{1}{2}\int\left(\frac{(-\cos x)'}{1+\cos x}+\frac{(-\cos x)'}{1-\cos x}\right)dx\\ &= \frac{1}{2}\left(-\log|1+\cos x|+\log|1-\cos x|\right)\\ &= \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right|\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\cos x}\,dx &= \int \frac{\cos x}{\cos^2 x}\,dx\\ &= \int \frac{\cos x}{1-\sin^2 x}\,dx\\ &= \frac{1}{2}\int\left(\frac{(\sin x)'}{1+\sin x}+\frac{(\sin x)'}{1-\sin x}\right)dx\\ &= \frac{1}{2}\left(\log|1+\sin x|-\log|1-\sin x|\right)\\ &= \frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right|\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sin^2 x}\,dx &= \int \frac{1}{\sin^2 x}\,dx\\ &= \int \frac{\cos^2 x}{\sin^2 x}\frac{1}{\cos^2 x}\,dx\\ &= \int \frac{1}{\tan^2 x}(\tan x)'\,dx\\ &= -\frac{1}{\tan x}\end{aligned}$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x$$

$$\begin{aligned}\int \frac{1}{\sin^3 x}\,dx &= \int \frac{\sin x}{\sin^4 x}\,dx\\ &= \int \frac{\sin x}{(1-\cos^2 x)^2}\,dx\\ &= \frac{1}{4}\int\left(\frac{(-\cos x)'}{1+\cos x}+\frac{(-\cos x)'}{1-\cos x}\right.\\ &\qquad\left.+\frac{(-\cos x)'}{(1+\cos x)^2}+\frac{(-\cos x)'}{(1-\cos x)^2}\right)dx\\ &= \frac{1}{4}\left(-\log|1+\cos x|+\log|1-\cos x|\right.\\ &\qquad\left.+\frac{1}{1+\cos x}-\frac{1}{1-\cos x}\right)\\ &= \frac{1}{4}\left(\log\left|\frac{1-\cos x}{1+\cos x}\right|+\frac{-2\cos x}{1-\cos^2 x}\right)\\ &= \frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right|-\frac{1}{2}\frac{\cos x}{\sin^2 x}\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\cos^3 x}\,dx &= \int \frac{\cos x}{\cos^4 x}\,dx\\ &= \int \frac{\cos x}{(1-\sin^2 x)^2}\,dx\\ &= \frac{1}{4}\int\left(\frac{(\sin x)'}{1+\sin x}+\frac{(\sin x)'}{1-\sin x}\right.\\ &\qquad\left.+\frac{(\sin x)'}{(1+\sin x)^2}+\frac{(\sin x)'}{(1-\sin x)^2}\right)dx\\ &= \frac{1}{4}\left(\log|1+\sin x|-\log|1-\sin x|\right.\\ &\qquad\left.-\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right)\\ &= \frac{1}{4}\left(\log\left|\frac{1+\sin x}{1-\sin x}\right|+\frac{2\sin x}{1-\sin^2 x}\right)\\ &= \frac{1}{4}\log\left|\frac{1+\sin x}{1-\sin x}\right|+\frac{1}{2}\frac{\sin x}{\cos^2 x}\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sin^4 x}\,dx &= \int \frac{(\sin^2 x+\cos^2 x)\cos^2 x}{\sin^4 x}\frac{1}{\cos^2 x}\,dx\\ &= \int\left(\frac{\cos^2 x}{\sin^2 x}+\frac{\cos^4 x}{\sin^4 x}\right)(\tan x)'\,dx\\ &= \int\left(\frac{1}{\tan^2 x}+\frac{1}{\tan^4 x}\right)(\tan x)'\,dx\\ &= -\frac{1}{\tan x}-\frac{1}{3\tan^3 x}\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\cos^4 x}\,dx &= \int \frac{1}{\cos^2 x}\frac{1}{\cos^2 x}\,dx\\ &= \int (1+\tan^2 x)(\tan x)'\,dx\\ &= \tan x+\frac{\tan^3 x}{3}\end{aligned}$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{(-\cos x)'}{\cos x}\,dx = -\log|\cos x|$$

$$\int \frac{1}{\tan x}\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{(\sin x)'}{\sin x}\,dx = \log|\sin x|$$

$$\int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} - 1\right)dx = \tan x - x$$

$$\int \frac{1}{\tan^2 x}\,dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx = \int \left(\frac{1}{\sin^2 x} - 1\right)dx = -\frac{1}{\tan x} - x$$

$$\int \tan^3 x\,dx = \int \left(\tan x(\tan^2 x + 1) - \tan x\right)dx = \int \left(\tan x(\tan x)' - \tan x\right)dx = \frac{\tan^2 x}{2} + \log|\cos x|$$

$$\int \frac{1}{\tan^3 x}\,dx = \int \left(\frac{1+\tan^2 x}{\tan^3 x} - \frac{1}{\tan x}\right)dx = \int \left(\frac{(\tan x)'}{\tan^3 x} - \frac{1}{\tan x}\right)dx = -\frac{1}{2\tan^2 x} - \log|\sin x|$$

逆三角関数

$$\int \sin^{-1} x\,dx \qquad \left(\text{部分積分} \quad \begin{cases} u' = 1 \\ v = \sin^{-1} x \end{cases} \rightarrow \begin{cases} u = x \\ v' = \dfrac{1}{\sqrt{1-x^2}} \end{cases}\right)$$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1} x + \sqrt{1-x^2}$$

$$\int \tan^{-1} x\,dx \qquad \left(\text{部分積分} \quad \begin{cases} u' = 1 \\ v = \tan^{-1} x \end{cases} \rightarrow \begin{cases} u = x \\ v' = \dfrac{1}{1+x^2} \end{cases}\right)$$

$$= x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\log\left|1+x^2\right|$$

分数式

$$\int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} \qquad (a \neq 0)$$

$$\int \frac{1}{(ax+b)^n}\,dx = -\frac{1}{a(n-1)(ax+b)^{n-1}} \qquad (a \neq 0,\ n \geqq 2)$$

$$\int \frac{1}{(x-\alpha)^2}\,dx = -\frac{1}{x-\alpha}$$

$$\begin{aligned}\int \frac{1}{(x-\alpha)(x-\beta)}\,dx &= \frac{1}{\alpha-\beta}\int\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right)dx = \frac{1}{\alpha-\beta}\left(\log|x-\alpha|-\log|x-\beta|\right)\\ &= \frac{1}{\alpha-\beta}\log\left|\frac{x-\alpha}{x-\beta}\right| \qquad (\alpha \neq \beta)\end{aligned}$$

$$\int \frac{1}{x^2-a^2}\,dx = \int \frac{1}{(x-a)(x+a)}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| \qquad (a \neq 0)$$

$$\begin{aligned}&\int \frac{1}{x^2+a^2}\,dx \qquad \left(\text{置換積分}\quad x = a\tan\theta\right)\\ &= \int \frac{1}{a^2(\tan^2\theta+1)}\,a(1+\tan^2\theta)d\theta = \int \frac{1}{a}\,d\theta = \frac{\theta}{a} = \frac{1}{a}\tan^{-1}\frac{x}{a} \qquad (a \neq 0)\end{aligned}$$

$$\int \frac{1}{(x-p)^2+q^2}\,dx = \frac{1}{q}\tan^{-1}\frac{x-p}{q} \qquad (q \neq 0)$$

$$\int \frac{2ax+b}{ax^2+bx+c}\,dx = \int \frac{(ax^2+bx+c)'}{ax^2+bx+c}\,dx = \log\left|ax^2+bx+c\right|$$

$$\begin{aligned}\int \frac{dx+e}{ax^2+bx+c}\,dx &= \int \frac{D(2ax+b)+E}{ax^2+bx+c}\,dx\\ &= D\log\left|ax^2+bx+c\right| + \begin{cases}\displaystyle\int \frac{E}{a(x-\alpha)(x-\beta)}\,dx & (b^2-4ac>0 \quad \text{のとき})\\ \displaystyle\int \frac{E}{a(x-\alpha)^2}\,dx & (b^2-4ac=0 \quad \text{のとき})\\ \displaystyle\int \frac{E}{a\left((x-p)^2+q^2\right)}\,dx & (b^2-4ac<0 \quad \text{のとき})\end{cases}\end{aligned}$$

と変形する

根号を含む式

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx \qquad (\text{置換積分}\quad x=a\sin\theta)$$
$$=\int \frac{1}{a\cos\theta}\,a\cos\theta d\theta$$
$$=\int 1\,d\theta=\theta$$
$$=\sin^{-1}\frac{x}{a}\qquad (a>0)$$

$$\int \frac{1}{\sqrt{x^2+A}}\,dx \qquad \left(\text{置換積分}\quad x=\frac{1}{2}\left(t-\frac{A}{t}\right)\right)$$
$$=\int \frac{1}{\frac{1}{2}\left(t+\frac{A}{t}\right)}\,\frac{1}{2}\left(1+\frac{A}{t^2}\right)dt$$
$$=\int \frac{1}{t}\,dt=\log|t|$$
$$=\log\left|\left(x+\sqrt{x^2+A}\right)\right|$$

$$\int \frac{x}{\sqrt{a^2-x^2}}\,dx=\int \frac{-\frac{1}{2}(a^2-x^2)'}{\sqrt{a^2-x^2}}\,dx$$
$$=-\sqrt{a^2-x^2}$$

$$\int \frac{x}{\sqrt{x^2+A}}\,dx=\int \frac{\frac{1}{2}(x^2+A)'}{\sqrt{x^2+A}}\,dx$$
$$=\sqrt{x^2+A}$$

$$\int \sqrt{a^2-x^2}\,dx \qquad (\text{置換積分}\quad x=a\sin\theta)$$
$$=\int a\cos\theta\,a\cos\theta d\theta$$
$$=a^2\int \cos^2\theta\,d\theta$$
$$=a^2\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)$$
$$=a^2\frac{\theta}{2}+\frac{a\sin\theta\,a\cos\theta}{2}$$
$$=\frac{1}{2}\left(a^2\sin^{-1}\frac{x}{a}+x\sqrt{a^2-x^2}\right)$$

$$\int \sqrt{x^2+A}\,dx \qquad \left(\text{置換積分}\quad x=\frac{1}{2}\left(t-\frac{A}{t}\right)\right)$$
$$=\int \frac{1}{2}\left(t+\frac{A}{t}\right)\frac{1}{2}\left(1+\frac{A}{t^2}\right)dt$$
$$=\frac{1}{4}\int\left(t+\frac{2A}{t}+\frac{A^2}{t^3}\right)dx$$
$$=\frac{1}{4}\left(\frac{t^2}{2}+2A\log|t|-\frac{A^2}{2t^2}\right)$$
$$=\frac{1}{2}\left(\frac{1}{2}\left(t-\frac{A}{t}\right)\frac{1}{2}\left(t+\frac{A}{t}\right)+A\log|t|\right)$$
$$=\frac{1}{2}\left(x\sqrt{x^2+A}+A\log\left|x+\sqrt{x^2+A}\right|\right)$$

$$\int \sqrt{a^2-x^2}\,dx=\int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx$$
$$=\int x\cdot\frac{-x}{\sqrt{a^2-x^2}}\,dx$$
$$+\int \frac{a^2}{\sqrt{a^2-x^2}}\,dx$$
$$=x\sqrt{a^2-x^2}-\int 1\sqrt{a^2-x^2}\,dx$$
$$+a^2\sin^{-1}\frac{x}{a}$$

$$\int \sqrt{x^2+A}\,dx=\int \frac{x^2+A}{\sqrt{x^2+A}}\,dx$$
$$=\int x\cdot\frac{x}{\sqrt{x^2+A}}\,dx$$
$$+\int \frac{A}{\sqrt{x^2+A}}\,dx$$
$$=x\sqrt{x^2+A}-\int 1\sqrt{x^2+A}\,dx$$
$$+A\log\left|x+\sqrt{x^2+A}\right|$$

ゆえに

$$2\int \sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}$$
$$\int \sqrt{a^2-x^2}\,dx=\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right)$$

ゆえに

$$2\int \sqrt{x^2+A}\,dx=x\sqrt{x^2+A}+A\log\left|x+\sqrt{x^2+A}\right|$$
$$\int \sqrt{x^2+A}\,dx=\frac{1}{2}\left(x\sqrt{x^2+A}+A\log\left|x+\sqrt{x^2+A}\right|\right)$$

指数・対数関数

$$\int a^x\ dx = \int e^{x\log a}\ dx = \frac{e^{x\log a}}{\log a} = \frac{a^x}{\log a}$$

$$\int xe^x\ dx \qquad \left(\text{部分積分}\ \begin{cases} u = x \\ v' = e^x \end{cases} \to \begin{cases} u' = 1 \\ v = e^x \end{cases}\right)$$

$$= xe^x - \int e^x\ dx = xe^x - e^x = (x-1)e^x$$

$$\int xe^{x^2}\ dx = \int \frac{1}{2}e^{x^2}(x^2)'\ dx = \frac{1}{2}e^{x^2}$$

$$\int \log x\ dx \qquad \left(\text{部分積分}\ \begin{cases} u' = 1 \\ v = \log x \end{cases} \to \begin{cases} u = x \\ v' = \dfrac{1}{x} \end{cases}\right)$$

$$= x\log x - \int 1\ dx = x\log x - x$$

$$\int (\log x)^2\ dx \qquad \left(\text{部分積分}\ \begin{cases} u' = 1 \\ v = (\log x)^2 \end{cases} \to \begin{cases} u = x \\ v' = \dfrac{2\log x}{x} \end{cases}\right)$$

$$= x(\log x)^2 - \int 2\log x\ dx = x(\log x)^2 - 2x\log x + 2x$$

$$\int x\log x\ dx \qquad \left(\text{部分積分}\ \begin{cases} u' = x \\ v = \log x \end{cases} \to \begin{cases} u = \dfrac{x^2}{2} \\ v' = \dfrac{1}{x} \end{cases}\right)$$

$$= \frac{x^2\log x}{2} - \int \frac{x}{2}\ dx = \frac{x^2\log x}{2} - \frac{x^2}{4}$$

$$\int \frac{\log x}{x}\ dx = \int \log x(\log x)'\ dx = \frac{(\log x)^2}{2}$$

$$\int \log\left(x+\sqrt{x^2+A}\right)\ dx \qquad \left(\text{部分積分}\begin{cases} u' = 1 \\ v = \log\left(x+\sqrt{x^2+A}\right) \end{cases} \to \begin{cases} u = x \\ v' = \dfrac{1}{\sqrt{x^2+A}} \end{cases}\right)$$

$$= x\log\left(x+\sqrt{x^2+A}\right) - \int \frac{x}{\sqrt{x^2+A}}\ dx = x\log\left(x+\sqrt{x^2+A}\right) - \sqrt{x^2+A}$$

双曲線関数

$$\int \cosh x\, dx = \sinh x$$

$$\int \sinh x\, dx = \cosh x$$

$$\int \frac{e^x + e^{-x}}{2}\, dx = \frac{e^x - e^{-x}}{2}$$

$$\int \frac{e^x - e^{-x}}{2}\, dx = \frac{e^x + e^{-x}}{2}$$

$$\begin{aligned}\int \cosh^2 x\, dx &= \int \frac{\cosh 2x + 1}{2}\, dx \\ &= \frac{\sinh 2x}{4} + \frac{x}{2}\end{aligned}$$

$$\begin{aligned}\int \sinh^2 x\, dx &= \int \frac{\cosh 2x - 1}{2}\, dx \\ &= \frac{\sinh 2x}{4} - \frac{x}{2}\end{aligned}$$

$$\begin{aligned}\int \left(\frac{e^x + e^{-x}}{2}\right)^2 dx &= \int \frac{e^{2x} + e^{-2x} + 2}{4}\, dx \\ &= \frac{e^{2x} - e^{-2x}}{8} + \frac{x}{2}\end{aligned}$$

$$\begin{aligned}\int \left(\frac{e^x - e^{-x}}{2}\right)^2 dx &= \int \frac{e^{2x} + e^{-2x} - 2}{4}\, dx \\ &= \frac{e^{2x} - e^{-2x}}{8} - \frac{x}{2}\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\cosh x}\, dx &= \int \frac{\cosh x}{\cosh^2 x}\, dx \\ &= \int \frac{(\sinh x)'}{\sinh^2 x + 1}\, dx \\ &= \tan^{-1} \sinh x\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sinh x}\, dx &= \int \frac{\sinh x}{\sinh^2 x}\, dx \\ &= \int \frac{(\cosh x)'}{\cosh^2 x - 1}\, dx \\ &= \frac{1}{2} \log \left| \frac{\cosh x - 1}{\cosh x + 1} \right|\end{aligned}$$

$$\begin{aligned}\int \frac{2}{e^x + e^{-x}}\, dx &= \int \frac{2e^x}{(e^x)^2 + 1}\, dx \\ &= 2 \tan^{-1} e^x\end{aligned}$$

$$\begin{aligned}\int \frac{2}{e^x - e^{-x}}\, dx &= \int \frac{2e^x}{(e^x)^2 - 1}\, dx \\ &= \log \left| \frac{e^x - 1}{e^x + 1} \right|\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\cosh^2 x}\, dx &= \int \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}\, dx \\ &= \frac{\sinh x}{\cosh x} \\ &= \tanh x\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sinh^2 x}\, dx &= \int \frac{-(\sinh^2 x - \cosh^2 x)}{\sinh^2 x}\, dx \\ &= \frac{-\cosh x}{\sinh x} \\ &= -\frac{1}{\tanh x}\end{aligned}$$

$$\begin{aligned}\int \left(\frac{2}{e^x + e^{-x}}\right)^2 dx &= \int \frac{2(2e^{2x})}{(e^{2x} + 1)^2}\, dx \\ &= -\frac{2}{e^{2x} + 1}\end{aligned}$$

$$\begin{aligned}\int \left(\frac{2}{e^x - e^{-x}}\right)^2 dx &= \int \frac{2(2e^{2x})}{(e^{2x} - 1)^2}\, dx \\ &= -\frac{2}{e^{2x} - 1}\end{aligned}$$

$$\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx = \log|\cosh x|$$

$$\int \frac{1}{\tanh x}\,dx = \int \frac{\cosh x}{\sinh x}\,dx = \log|\sinh x|$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \log\left|e^x + e^{-x}\right|$$

$$\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \log\left|e^x - e^{-x}\right|$$

$$\int \tanh^2 x\,dx = \int \big(-(1-\tanh^2 x) + 1\big)\,dx = -\tanh x + x$$

$$\int \frac{1}{\tanh^2 x}\,dx = \int \left(\frac{1-\tanh^2 x}{\tanh^2 x} + 1\right)dx = -\frac{1}{\tanh x} + x$$

$$\begin{aligned}\int \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 dx &= \int \left(\frac{e^{2x}-1}{e^{2x}+1}\right)^2 dx \\ &= \int \frac{\left(e^{2x}+1\right)^2 - 4e^{2x}}{\left(e^{2x}+1\right)^2}\,dx \\ &= \int \left(1 - \frac{2(2e^{2x})}{(e^{2x}+1)^2}\right)dx \\ &= x + \frac{2}{e^{2x}+1}\end{aligned}$$

$$\begin{aligned}\int \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 dx &= \int \left(\frac{e^{2x}+1}{e^{2x}-1}\right)^2 dx \\ &= \int \frac{\left(e^{2x}-1\right)^2 + 4e^{2x}}{\left(e^{2x}-1\right)^2}\,dx \\ &= \int \left(1 + \frac{2(2e^{2x})}{(e^{2x}-1)^2}\right)dx \\ &= x - \frac{2}{e^{2x}-1}\end{aligned}$$

その他の不定積分

$$\begin{aligned}
C(x) &= \int e^{ax}\cos bx\;dx \\
&= \frac{e^{ax}}{a}\cos bx - \int \frac{e^{ax}}{a}(-b\sin bx)\;dx \\
&= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}S(x) \\
&= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\left(\frac{e^{ax}}{a}\sin bx - \frac{b}{a}C(x)\right) \\
a^2C(x) &= e^{ax}(a\cos bx + b\sin bx) - b^2C(x)
\end{aligned}$$

$$\begin{aligned}
(a^2+b^2)C(x) &= e^{ax}(a\cos bx + b\sin bx) \\
C(x) &= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)
\end{aligned}$$

$$\begin{aligned}
S(x) &= \int e^{ax}\sin bx\;dx \\
&= \frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax}}{a}(b\cos bx)\;dx \\
&= \frac{e^{ax}}{a}\sin bx - \frac{b}{a}C(x) \\
&= \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left(\frac{e^{ax}}{a}\cos bx + \frac{b}{a}S(x)\right) \\
a^2S(x) &= e^{ax}(a\sin bx - b\cos bx) - b^2S(x)
\end{aligned}$$

$$\begin{aligned}
(a^2+b^2)S(x) &= e^{ax}(a\sin bx - b\cos bx) \\
S(x) &= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)
\end{aligned}$$

定積分

$$\int_\alpha^\beta (x-\alpha)(x-\beta)\,dx = \left[\ \frac{(x-\alpha)^2}{2}\,(x-\beta)\ \right]_\alpha^\beta - a\int_\alpha^\beta \frac{(x-\alpha)^2}{2}\cdot 1\,dx$$
$$= -\left[\ \frac{(x-\alpha)^3}{6}\ \right]_\alpha^\beta$$
$$= -\frac{(\beta-\alpha)^3}{6}$$

$$\int_\alpha^\beta (x-\alpha)(x-\beta)\,dx = \int_\alpha^\beta (x-\alpha)\left((x-\alpha)-(\beta-\alpha)\right)dx$$
$$= \int_\alpha^\beta \big((x-\alpha)^2-(\beta-\alpha)(x-\alpha)\big)\,dx$$
$$= \left[\ \frac{(x-\alpha)^3}{3} - (\beta-\alpha)\frac{(x-\alpha)^2}{2}\ \right]_\alpha^\beta$$
$$= \left(\frac{(\beta-\alpha)^3}{3} - \frac{(\beta-\alpha)^3}{2}\right)$$
$$= -\frac{(\beta-\alpha)^3}{6}$$

$$\int_\alpha^\gamma (x-\alpha)(x-\beta)(x-\gamma)\,dx = \int_\alpha^\gamma (x-\alpha)\left((x-\alpha)-(\beta-\alpha)\right)\left((x-\alpha)-(\gamma-\alpha)\right)dx$$
$$= \int_\alpha^\gamma \big((x-\alpha)^3 - \left((\beta-\alpha)+(\gamma-\alpha)\right)(x-\alpha)^2 + (\beta-\alpha)(\gamma-\alpha)(x-\alpha)\big)\,dx$$
$$= \left[\ \frac{(x-\alpha)^4}{4} - \left((\beta-\alpha)+(\gamma-\alpha)\right)\frac{(x-\alpha)^3}{3} + (\beta-\alpha)(\gamma-\alpha)\frac{(x-\alpha)^2}{2}\ \right]_\alpha^\gamma$$
$$= \frac{(\gamma-\alpha)^3}{12}\left(3(\gamma-\alpha)-4(\beta-\alpha)-4(\gamma-\alpha)+6(\beta-\alpha)\right)$$
$$= -\frac{(\gamma-\alpha)^3(\gamma-2\beta+\alpha)}{12}$$

$$\int_\alpha^\beta (x-\alpha)^n(x-\beta)\,dx = \left[\ \frac{(x-\alpha)^{n+1}}{n+1}(x-\beta)\ \right]_\alpha^\beta - \int_\alpha^\beta \frac{(x-\alpha)^{n+1}}{n+1}\cdot 1\,dx$$
$$= -\left[\ \frac{(x-\alpha)^{n+2}}{(n+2)(n+1)}\ \right]_\alpha^\beta$$
$$= -\frac{(\beta-\alpha)^{n+2}}{(n+2)(n+1)}$$

$$
\begin{aligned}
S_{m,n} &= \int_\alpha^\beta (x-\alpha)^m (x-\beta)^n \, dx \\
&= \left[\frac{(x-\alpha)^{m+1}}{m+1} (x-\beta)^n \right]_\alpha^\beta - \int_\alpha^\beta \frac{(x-\alpha)^{m+1}}{m+1} \cdot n(x-\beta)^{n-1} \, dx \\
&= -\frac{n}{m+1} S_{m+1,n-1} \\
&= +\frac{n}{m+1} \frac{n-1}{m+2} S_{m+2,n-2} \\
&= \cdots \\
&= (-1)^n \frac{n(n-1)\cdots 1}{(m+1)(m+2)\cdots(m+n)} S_{m+n,0} \\
&= (-1)^n \frac{m!\,n!}{(m+n)!} \int_\alpha^\beta (x-\alpha)^{m+n} \, dx \\
&= (-1)^n \frac{m!\,n!}{(m+n+1)!} (\beta-\alpha)^{m+n+1}
\end{aligned}
$$

$$
\begin{aligned}
S_n &= \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \sin x \sin^{n-1} x \, dx \\
&= \left[-\cos x \sin^{n-1} x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x (n-1) \sin^{n-2} x \cos x \, dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x) \, dx \\
&= \int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x \, dx \\
&= (n-1) S_{n-2} - (n-1) S_n
\end{aligned}
$$

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$$
\begin{aligned}
S_n &= \frac{n-1}{n} S_{n-2} = \frac{n-1}{n} \frac{n-3}{n-2} S_{n-4} = \cdots \\
&= \begin{cases} \dfrac{n-1}{n} \dfrac{n-3}{n-2} \cdots \dfrac{1}{2} \, S_0 & (n : \text{偶数}) \\[2ex] \dfrac{n-1}{n} \dfrac{n-3}{n-2} \cdots \dfrac{2}{3} \, S1 & (n : \text{奇数}) \end{cases} \\
&= \begin{cases} \dfrac{n-1}{n} \dfrac{n-3}{n-2} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & (n : \text{偶数}) \\[2ex] \dfrac{n-1}{n} \dfrac{n-3}{n-2} \cdots \dfrac{2}{3} \cdot 1 & (n : \text{奇数}) \end{cases}
\end{aligned}
$$

同様に

$$
\begin{aligned}
C_n &= \int_0^{\frac{\pi}{2}} \cos^n x \, dx \\
&= \begin{cases} \dfrac{n-1}{n} \dfrac{n-3}{n-2} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & (n : \text{偶数}) \\[2ex] \dfrac{n-1}{n} \dfrac{n-3}{n-2} \cdots \dfrac{2}{3} \cdot 1 & (n : \text{奇数}) \end{cases}
\end{aligned}
$$