公式（指数・対数関数）

指数の定義

自然数 m について

$$a^m = \overbrace{a \times a \times \cdots \times a}^{m\text{ 個}}$$

整数に拡張

$$a^{m-n} = \frac{a^m}{a^n} \qquad a^0 = 1 \qquad a^{-n} = \frac{1}{a^n}$$

有理数に拡張（$a > 0$）

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \qquad a^{\frac{1}{n}} = \sqrt[n]{a}$$

無理数に拡張（$a > 0$）

$1 < a$ の場合 $\quad a^x = y \Leftrightarrow \forall\, p, q$: 有理数（$p < x < q \ \rightarrow\ a^p < y < a^q$）

$0 < a < 1$ の場合 $\quad a^x = y \Leftrightarrow \forall\, p, q$: 有理数（$p < x < q \ \rightarrow\ a^p > y > a^q$）

指数法則

$$a^{x+y} = a^x \cdot a^y \qquad a^{x-y} = \frac{a^x}{a^y}$$

$$a^{x \cdot y} = (a^x)^y = (a^y)^x \qquad a^{\frac{x}{y}} = (a^x)^{\frac{1}{y}} = \left(a^{\frac{1}{y}}\right)^x$$

$$(a \cdot b)^x = a^x \cdot b^x \qquad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

指数関数の大小関係

$1 < a$ の場合 ： $x < y \iff a^x < a^y$

$0 < a < 1$ の場合 ： $x < y \iff a^x > a^y$

対数の定義

$x = \log_a X \iff X = a^x$

$\log_a a^x = x$　　　$a^{\log_a X} = X$

対数法則

$\log_a(X \cdot Y) = \log_a X + \log_a Y$　　　$\log_a \dfrac{X}{Y} = \log_a X - \log_a Y$

$\log_a Y^x = x \log_a Y$

底の変換公式

$$\log_X Y = \frac{\log_a Y}{\log_a X}$$

対数関数の大小関係

$1 < a$ の場合 ： $x < y \iff \log_a x < \log_a y$

$0 < a < 1$ の場合 ： $x < y \iff \log_a x > \log_a y$

常用対数

$\log_{10} 2 = 0.30103\ldots$　　　$\log_{10} 3 = 0.47712\ldots$

$\log_{10} 4 = 0.60206\ldots\ (= 2\log_{10} 2)$　　　$\log_{10} 5 = 0.69897\ldots\ (= 1 - \log_{10} 2)$

$\log_{10} 6 = 0.77815\ldots\ (= \log_{10} 2 + \log_{10} 3)$　　　$\log_{10} 7 = 0.84509\ldots$

$\log_{10} 8 = 0.90309\ldots\ (= 3\log_{10} 2)$　　　$\log_{10} 9 = 0.95424\ldots\ (= 2\log_{10} 3)$

桁数と最高位の数字

M が n 桁の数 $\iff 10^{n-1} \leqq M < 10^n$

$\iff n = (\log_{10} M$ の整数部分$) + 1$

最高位の数字が a $\iff a \times 10^{n-1} \leqq M < (a+1) \times 10^{n-1}$

$\iff \log_{10} a \leqq (\log_{10} M$ の小数部分$) < \log_{10}(a+1)$

Napier の定数 e の定義

$$e = \left(\lim_{t \to 0} \frac{a^t - 1}{t} = 1 \text{ となる } a \right) \qquad \lim_{t \to 0} \frac{e^t - 1}{t} = 1$$

e の定義と同値な関係

$$e = \left(\lim_{t \to 0} \frac{\log_a (t+1)}{t} = 1 \text{ となる } a \right) \qquad \lim_{t \to 0} \frac{\log_e (t+1)}{t} = 1$$

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e \qquad \lim_{n \to -\infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e \qquad \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

自然対数の定義

$\log x = \log_e x$ 　(底 e を省略する)　 $\ln x$ と書くこともある

双曲線関数

$$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^\theta - e^{-\theta}}{2} \qquad \tanh\theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}$$

相互関係

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta} \qquad \cosh^2\theta - \sinh^2\theta = 1 \qquad 1 - \tanh^2\theta = \frac{1}{\cosh^2\theta}$$

加法定理

$$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta \qquad \cosh(\alpha-\beta) = \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta$$

$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta \qquad \sinh(\alpha-\beta) = \sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta$$

$$\tanh(\alpha+\beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} \qquad \tanh(\alpha-\beta) = \frac{\tanh\alpha - \tanh\beta}{1 - \tanh\alpha\tanh\beta}$$

逆双曲線関数

$$\theta = \cosh^{-1} x \iff \cosh\theta = x \text{ かつ } \theta \geqq 0 \qquad \cosh^{-1} x = \log\left(x + \sqrt{x^2 - 1}\right)$$

$$\theta = \sinh^{-1} y \iff \sinh\theta = y \qquad \sinh^{-1} y = \log\left(y + \sqrt{y^2 + 1}\right)$$

$$\theta = \tanh^{-1} m \iff \tanh\theta = m \qquad \tanh^{-1} m = \frac{1}{2}\log\frac{1+m}{1-m}$$

証明

対数法則

$\log_a X = x,\ \log_a Y = y$　とおく

$$X = a^x,\ Y = a^y$$

$$X \cdot Y = a^x \cdot a^y = a^{x+y}$$
$$\therefore\ \log_a (X \cdot Y) = x + y = \log_a X + \log_a Y$$

$$\frac{X}{Y} = \frac{a^x}{a^y} = a^{x-y}$$
$$\therefore\ \log_a \frac{X}{Y} = x - y = \log_a X - \log_a Y$$

$$Y^x = (a^y)^x = a^{xy}$$
$$\therefore\ \log_a Y^x = xy = x \log_a Y$$

底の変換

$$Y = a^y = a^{x \cdot \frac{y}{x}} = (a^x)^{\frac{y}{x}} = X^{\frac{y}{x}}$$
$$\therefore\ \log_X Y = \frac{y}{x} = \frac{\log_a Y}{\log_a X}$$

e の定義と同値な関係

- $\displaystyle \lim_{x\to 0}\frac{e^x-1}{x}=1 \quad \Rightarrow \quad \lim_{t\to 0}\frac{\log_e(t+1)}{t}=1$ （逆も同様）

$x=\log_e(t+1)$ とおく

$t=e^x-1$

$t\to 0$ のとき $x\to\log_e 1=0$

$$\therefore \lim_{t\to 0}\frac{\log_e(t+1)}{t}=\lim_{x\to 0}\frac{x}{e^x-1}=\frac{1}{\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}}=\frac{1}{1}=1$$

- $\displaystyle \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e \quad \Rightarrow \quad \lim_{t\to -\infty}\left(1+\frac{1}{t}\right)^t=e$ （逆も同様）

$x=-t-1$ とおく

$t=-x-1$

$t\to-\infty$ のとき $x\to\infty$

$$\therefore \lim_{t\to-\infty}\left(1+\frac{1}{t}\right)^t=\lim_{x\to\infty}\left(1-\frac{1}{x+1}\right)^{-x-1}=\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{-x-1}$$
$$=\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x+1}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\left(1+\frac{1}{x}\right)=e\cdot 1=1$$

- $\displaystyle \lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x=e \quad \Rightarrow \quad \lim_{t\to 0}\frac{\log_e(t+1)}{t}=1$ （逆も同様）

$x=\dfrac{1}{t}$ とおく

$t=\dfrac{1}{x}$

$t\to 0$ のとき $x\to\pm\infty$

$$\therefore \lim_{t\to 0}\frac{\log_e(t+1)}{t}=\lim_{x\to\pm\infty}\frac{\log_e\left(\dfrac{1}{x}+1\right)}{\dfrac{1}{x}}=\lim_{x\to\pm\infty}x\log_e\left(1+\frac{1}{x}\right)$$
$$=\lim_{x\to\pm\infty}\log_e\left(1+\frac{1}{x}\right)^x=\log_e e=1$$

- $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad\Rightarrow\quad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$ （逆は明らか）

$n=[x]=(x$ の整数部分$),\ m=n+1$ とする

$$n \leqq x < m$$
$$1+\frac{1}{m} < 1+\frac{1}{x} \leqq 1+\frac{1}{n}$$
$$\left(1+\frac{1}{m}\right)^n \leqq \left(1+\frac{1}{m}\right)^x < \left(1+\frac{1}{x}\right)^x \leqq \left(1+\frac{1}{n}\right)^x < \left(1+\frac{1}{n}\right)^m$$

$x\to\infty$ のとき $n\to\infty$, $m\to\infty$

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m-1} = \lim_{m\to\infty}\frac{\left(1+\frac{1}{m}\right)^m}{\left(1+\frac{1}{m}\right)} = \frac{e}{1} = e$$
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^m = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) = e\cdot 1 = e$$
$$\therefore\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

双曲線関数の相互関係

$$\tanh\theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}} = \frac{2\sinh\theta}{2\cosh\theta} = \frac{\sinh\theta}{\cosh\theta}$$

$$\cosh^2\theta - \sinh^2\theta = (\cosh\theta+\sinh\theta)(\cosh\theta-\sinh\theta) = e^\theta e^{-\theta} = 1$$

$$1-\tanh^2\theta = 1-\frac{\sinh^2\theta}{\cosh^2\theta} = \frac{\cosh^2\theta-\sinh^2\theta}{\cosh^2\theta} = \frac{1}{\cosh^2\theta}$$

双曲線関数の加法定理

$$
\begin{aligned}
\cosh\alpha\cosh\beta &= \frac{e^{\alpha}+e^{-\alpha}}{2}\,\frac{e^{\beta}+e^{-\beta}}{2} = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}+e^{-\alpha+\beta}+e^{-\alpha-\beta}}{4} \\
&= \frac{e^{\alpha+\beta}+e^{-\alpha-\beta}}{4} + \frac{e^{\alpha-\beta}+e^{-\alpha+\beta}}{4} = \frac{\cosh(\alpha+\beta)}{2} + \frac{\cosh(\alpha-\beta)}{2} \\
\sinh\alpha\sinh\beta &= \frac{e^{\alpha}-e^{-\alpha}}{2}\,\frac{e^{\beta}-e^{-\beta}}{2} = \frac{e^{\alpha+\beta}-e^{\alpha-\beta}-e^{-\alpha+\beta}+e^{-\alpha-\beta}}{4} \\
&= \frac{e^{\alpha+\beta}+e^{-\alpha-\beta}}{4} - \frac{e^{\alpha-\beta}+e^{-\alpha+\beta}}{4} = \frac{\cosh(\alpha+\beta)}{2} - \frac{\cosh(\alpha-\beta)}{2}
\end{aligned}
$$

ゆえに

$$
\begin{aligned}
\cosh(\alpha+\beta) &= \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta \\
\cosh(\alpha-\beta) &= \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta
\end{aligned}
$$

$$
\begin{aligned}
\sinh\alpha\cosh\beta &= \frac{e^{\alpha}-e^{-\alpha}}{2}\,\frac{e^{\beta}+e^{-\beta}}{2} = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}-e^{-\alpha+\beta}-e^{-\alpha-\beta}}{4} \\
&= \frac{e^{\alpha+\beta}-e^{-\alpha-\beta}}{4} + \frac{e^{\alpha-\beta}-e^{-\alpha+\beta}}{4} = \frac{\sinh(\alpha+\beta)}{2} + \frac{\sinh(\alpha-\beta)}{2} \\
\cosh\alpha\sinh\beta &= \frac{e^{\alpha}+e^{-\alpha}}{2}\,\frac{e^{\beta}-e^{-\beta}}{2} = \frac{e^{\alpha+\beta}-e^{\alpha-\beta}+e^{-\alpha+\beta}-e^{-\alpha-\beta}}{4} \\
&= \frac{e^{\alpha+\beta}-e^{-\alpha-\beta}}{4} - \frac{e^{\alpha-\beta}-e^{-\alpha+\beta}}{4} = \frac{\sinh(\alpha+\beta)}{2} - \frac{\sinh(\alpha-\beta)}{2}
\end{aligned}
$$

ゆえに

$$
\begin{aligned}
\sinh(\alpha+\beta) &= \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta \\
\sinh(\alpha-\beta) &= \sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta
\end{aligned}
$$

$$
\begin{aligned}
\tanh(\alpha+\beta) &= \frac{\sinh(\alpha+\beta)}{\cosh(\alpha+\beta)} = \frac{\sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta}{\cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta} \\
&= \frac{\dfrac{\sinh\alpha}{\cosh\alpha} + \dfrac{\sinh\beta}{\cosh\beta}}{1 + \dfrac{\sinh\alpha}{\cosh\alpha}\dfrac{\sinh\beta}{\cosh\beta}} = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} \\
\tanh(\alpha-\beta) &= \frac{\sinh(\alpha-\beta)}{\cosh(\alpha-\beta)} = \frac{\sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta}{\cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta} \\
&= \frac{\dfrac{\sinh\alpha}{\cosh\alpha} - \dfrac{\sinh\beta}{\cosh\beta}}{1 - \dfrac{\sinh\alpha}{\cosh\alpha}\dfrac{\sinh\beta}{\cosh\beta}} = \frac{\tanh\alpha - \tanh\beta}{1 - \tanh\alpha\tanh\beta}
\end{aligned}
$$

逆双曲線関数

$$\theta = \cosh^{-1} x \iff x = \cosh\theta \quad \text{かつ} \quad \theta \geqq 0$$

ゆえに

$$x = \frac{e^{\theta} + e^{-\theta}}{2} = \frac{\left(e^{\theta}\right)^2 + 1}{2e^{\theta}}$$

$$\left(e^{\theta}\right)^2 - 2xe^{\theta} + 1 = 0$$

$\theta \geqq 0$ より $e^{\theta} \geqq 1$ だから

$$e^{\theta} = x + \sqrt{x^2 - 1}$$

$$\therefore \quad \cosh^{-1} x = \theta = \log\left(x + \sqrt{x^2 - 1}\right)$$

$$\theta = \sinh^{-1} y \iff y = \sinh\theta$$

ゆえに

$$y = \frac{e^{\theta} - e^{-\theta}}{2} = \frac{\left(e^{\theta}\right)^2 - 1}{2e^{\theta}}$$

$$\left(e^{\theta}\right)^2 - 2ye^{\theta} - 1 = 0$$

$e^{\theta} > 0$ だから

$$e^{\theta} = y + \sqrt{y^2 + 1}$$

$$\therefore \quad \sinh^{-1} y = \theta = \log\left(y + \sqrt{y^2 + 1}\right)$$

$$\theta = \tanh^{-1} m \iff m = \tanh\theta$$

ゆえに

$$m = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} = \frac{e^{2\theta} - 1}{e^{2\theta} + 1}$$

$$m(e^{2\theta} + 1) = e^{2\theta} - 1$$

$$1 + m = (1 - m)e^{2\theta}$$

$$e^{2\theta} = \frac{1 + m}{1 - m}$$

$$2\theta = \log\frac{1 + m}{1 - m}$$

$$\therefore \quad \tanh^{-1} m = \theta = \frac{1}{2}\log\frac{1 + m}{1 - m}$$