2　形式演繹体系の証明例（述語編）

2.1　問題

(1) $\neg\exists x P(x) \iff \forall x \neg P(x)$

(2) $\neg\forall x P(x) \iff \exists x \neg P(x)$ （ $\Longrightarrow$ は古典）

(3) $\forall x \forall y P(x,y) \iff \forall y \forall x P(x,y)$

(4) $\exists x \exists y P(x,y) \iff \exists y \exists x P(x,y)$

(5) $\forall x P(x) \land \forall x Q(x) \iff \forall x (P(x) \land Q(x))$

(6) $\exists x P(x) \lor \exists x Q(x) \iff \exists x (P(x) \lor Q(x))$

(7) $\forall x P(x) \lor \forall x Q(x) \iff \forall x \forall y (P(x) \lor Q(y))$ （ $\Longleftarrow$ は古典）

(7') $\forall x P(x) \lor \forall x Q(x) \Longrightarrow \forall x (P(x) \lor Q(x))$ （ $\Longleftarrow$ は成り立たない）

(8) $\exists x P(x) \land \exists x Q(x) \iff \exists x \exists y (P(x) \land Q(y))$

(8') $\exists x P(x) \land \exists x Q(x) \Longleftarrow \exists x (P(x) \land Q(x))$ （ $\Longrightarrow$ は成り立たない）

(9) $\forall x P(x) \to \forall x Q(x) \iff \exists x \forall y (P(x) \to Q(y))$ （ $\Longrightarrow$ は古典）

(10) $\exists x P(x) \to \exists x Q(x) \iff \exists y \forall x (P(x) \to Q(y))$ （ $\Longrightarrow$ は古典）

(11) $\forall x P(x) \to \exists x Q(x) \iff \exists x (P(x) \to Q(x))$ （ $\Longrightarrow$ は古典）

(12) $\exists x P(x) \to \forall x Q(x) \iff \forall x \forall y (P(x) \to Q(y))$

(12') $\exists x P(x) \to \forall x Q(x) \Longrightarrow \forall x (P(x) \to Q(x))$ （ $\Longleftarrow$ は成り立たない）

(13) $\forall x \exists y (P(x) \to Q(y)) \iff \exists y \forall x (P(x) \to Q(y))$ （ $\Longrightarrow$ は古典）

(13') $\forall x \exists y P(x,y) \Longleftarrow \exists y \forall x P(x,y)$ （ $\Longrightarrow$ は成り立たない）

2.2 解答

(1) $\neg\exists xP(x) \iff \forall x\neg P(x)$

(1-1) $\neg\exists xP(x) \Longrightarrow \forall x\neg P(x)$

$$\begin{array}{c} P(a) \Longrightarrow P(a) \\ \hline P(a) \Longrightarrow \exists xP(x) \\ \hline P(a), \neg\exists xP(x) \Longrightarrow \\ \hline \neg\exists xP(x) \Longrightarrow \neg P(a) \\ \hline \neg\exists xP(x) \Longrightarrow \forall x\neg P(x) \end{array}$$

(1-2) $\forall x\neg P(x) \Longrightarrow \neg\exists xP(x)$

$$\begin{array}{c} P(a) \Longrightarrow P(a) \\ \hline P(a), \neg P(a) \Longrightarrow \\ \hline P(a), \forall x\neg P(x) \Longrightarrow \\ \hline \exists xP(x), \forall x\neg P(x) \Longrightarrow \\ \hline \forall x\neg P(x) \Longrightarrow \neg\exists xP(x) \end{array}$$

(2) $\neg\forall xP(x) \iff \exists x\neg P(x)$ （ $\Longrightarrow$ は古典）

(2-1) $\neg\forall xP(x) \Longrightarrow \exists x\neg P(x)$ (古典)

$$\begin{array}{c} P(a) \Longrightarrow P(a) \\ \hline \Longrightarrow \neg P(a), P(a) \\ \hline \Longrightarrow \exists x\neg P(x), P(a) \\ \hline \Longrightarrow \exists x\neg P(x), \forall xP(x) \\ \hline \neg\forall xP(x) \Longrightarrow \exists x\neg P(x) \end{array}$$

(2-2) $\exists x\neg P(x) \Longrightarrow \neg\forall xP(x)$

$$\begin{array}{c} P(a) \Longrightarrow P(a) \\ \hline \forall xP(x) \Longrightarrow P(a) \\ \hline \forall xP(x), \neg P(a) \Longrightarrow \\ \hline \neg P(a) \Longrightarrow \neg\forall xP(x) \\ \hline \exists x\neg P(x) \Longrightarrow \neg\forall xP(x) \end{array}$$

(3) $\forall x \forall y P(x,y) \iff \forall y \forall x P(x,y)$

(3-1) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$

$$\cfrac{\cfrac{\cfrac{\cfrac{P(a,b) \implies P(a,b)}{\forall y P(a,y) \implies P(a,b)}}{\forall x \forall y P(x,y) \implies P(a,b)}}{\forall x \forall y P(x,y) \implies \forall x P(x,b)}}{\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)}$$

(4) $\exists x \exists y P(x,y) \iff \exists y \exists x P(x,y)$

(4-1) $\exists x \exists y P(x,y) \implies \exists y \exists x P(x,y)$

$$\cfrac{\cfrac{\cfrac{\cfrac{P(a,b) \implies P(a,b)}{P(a,b) \implies \exists x P(x,b)}}{P(a,b) \implies \exists y \exists x P(x,y)}}{\exists y P(a,y) \implies \exists y \exists x P(x,y)}}{\exists x \exists y P(x,y) \implies \exists y \exists x P(x,y)}$$

(5) $\forall x P(x) \land \forall x Q(x) \iff \forall x (P(x) \land Q(x))$

(5-1) $\forall x P(x) \land \forall x Q(x) \implies \forall x (P(x) \land Q(x))$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \implies P(a)}{\forall x P(x) \implies P(a)}}{\forall x P(x) \land \forall x Q(x) \implies P(a)} \quad \dfrac{\dfrac{Q(a) \implies Q(a)}{\forall x Q(x) \implies Q(a)}}{\forall x P(x) \land \forall x Q(x) \implies Q(a)}}{\forall x P(x) \land \forall x Q(x) \implies P(a) \land Q(a)}}{\forall x P(x) \land \forall x Q(x) \implies \forall x (P(x) \land Q(x))}$$

(5-2) $\forall x (P(x) \land Q(x)) \implies \forall x P(x) \land \forall x Q(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \implies P(a)}{P(a) \land Q(a) \implies P(a)}}{\forall x (P(x) \land Q(x)) \implies P(a)}}{\forall x (P(x) \land Q(x)) \implies \forall x P(x)} \quad \dfrac{\dfrac{\dfrac{Q(b) \implies Q(b)}{P(b) \land Q(b) \implies Q(b)}}{\forall x (P(x) \land Q(x)) \implies Q(b)}}{\forall x (P(x) \land Q(x)) \implies \forall x Q(x)}}{\forall x (P(x) \land Q(x)) \implies \forall x P(x) \land \forall x Q(x)}$$

(6) $\exists x P(x) \lor \exists x Q(x) \iff \exists x (P(x) \lor Q(x))$

(6-1) $\exists x P(x) \lor \exists x Q(x) \implies \exists x (P(x) \lor Q(x))$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \implies P(a)}{P(a) \implies P(a) \lor Q(a)}}{P(a) \implies \exists x (P(x) \lor Q(x))}}{\exists x P(x) \implies \exists x (P(x) \lor Q(x))} \quad \dfrac{\dfrac{\dfrac{Q(b) \implies Q(b)}{Q(b) \implies P(b) \lor Q(b)}}{Q(b) \implies \exists x (P(x) \lor Q(x))}}{\exists x Q(x) \implies \exists x (P(x) \lor Q(x))}}{\exists x P(x) \lor \exists x Q(x) \implies \exists x (P(x) \lor Q(x))}$$

(6-2) $\exists x (P(x) \lor Q(x)) \implies \exists x P(x) \lor \exists x Q(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(a) \implies P(a)}{P(a) \implies \exists x P(x)}}{P(a) \implies \exists x P(x) \lor \exists x Q(x)} \quad \dfrac{\dfrac{Q(a) \implies Q(a)}{Q(a) \implies \exists x Q(x)}}{Q(a) \implies \exists x P(x) \lor \exists x Q(x)}}{P(a) \lor Q(a) \implies \exists x P(x) \lor \exists x Q(x)}}{\exists x (P(x) \lor Q(x)) \implies \exists x P(x) \lor \exists x Q(x)}$$

(7) $\forall x P(x) \vee \forall x Q(x) \iff \forall x \forall y (P(x) \vee Q(y))$ （ $\Longleftarrow$ は古典）

(7’) $\forall x P(x) \vee \forall x Q(x) \Longrightarrow \forall x (P(x) \vee Q(x))$ （ $\Longleftarrow$ は成り立たない）

(7-1) $\forall x P(x) \vee \forall x Q(x) \Longrightarrow \forall x \forall y (P(x) \vee Q(y))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Longrightarrow P(a)}{\forall x P(x) \Longrightarrow P(a)}}{\forall x P(x) \Longrightarrow P(a) \vee Q(b)} \quad \dfrac{\dfrac{Q(b) \Longrightarrow Q(b)}{\forall x Q(x) \Longrightarrow Q(b)}}{\forall x Q(x) \Longrightarrow P(a) \vee Q(b)}}{\forall x P(x) \vee \forall x Q(x) \Longrightarrow P(a) \vee Q(b)}}{\forall x P(x) \vee \forall x Q(x) \Longrightarrow \forall y (P(a) \vee Q(y))}}{\forall x P(x) \vee \forall x Q(x) \Longrightarrow \forall x \forall y (P(x) \vee Q(y))}}{}
$$

(7-2) $\forall x \forall y (P(x) \vee Q(y)) \Longrightarrow \forall x P(x) \vee \forall x Q(x)$ （古典）

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Longrightarrow P(a) \quad Q(b) \Longrightarrow Q(b)}{P(a) \vee Q(b) \Longrightarrow P(a), Q(b)}}{\forall y (P(a) \vee Q(y)) \Longrightarrow P(a), Q(b)}}{\forall x \forall y (P(x) \vee Q(y)) \Longrightarrow P(a), Q(b)}}{\forall x \forall y (P(x) \vee Q(y)) \Longrightarrow P(a), \forall x Q(x)}}{\forall x \forall y (P(x) \vee Q(y)) \Longrightarrow \forall x P(x), \forall x Q(x)}}{\forall x \forall y (P(x) \vee Q(y)) \Longrightarrow \forall x P(x) \vee \forall x Q(x)}
$$

(7’-1) $\forall x P(x) \vee \forall x Q(x) \Longrightarrow \forall x (P(x) \vee Q(x))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Longrightarrow P(a)}{\forall x P(x) \Longrightarrow P(a)}}{\forall x P(x) \Longrightarrow P(a) \vee Q(a)} \quad \dfrac{\dfrac{Q(a) \Longrightarrow Q(a)}{\forall x Q(x) \Longrightarrow Q(a)}}{\forall x Q(x) \Longrightarrow P(a) \vee Q(a)}}{\forall x P(x) \vee \forall x Q(x) \Longrightarrow P(a) \vee Q(a)}}{\forall x P(x) \vee \forall x Q(x) \Longrightarrow \forall x (P(x) \vee Q(x))}
$$

(8) $\exists x P(x) \land \exists x Q(x) \iff \exists x \exists y (P(x) \land Q(y))$

(8') $\exists x P(x) \land \exists x Q(x) \Longleftarrow \exists x (P(x) \land Q(x))$ （ $\Longrightarrow$ は成り立たない）

(8-1) $\exists x P(x) \land \exists x Q(x) \Longrightarrow \exists x \exists y (P(x) \land Q(y))$ (古典)

$$
\begin{array}{c}
P(a) \Longrightarrow P(a) \quad Q(b) \Longrightarrow Q(b) \\ \hline
P(a), Q(b) \Longrightarrow P(a) \land Q(b) \\ \hline
P(a), Q(b) \Longrightarrow \exists y (P(a) \land Q(y)) \\ \hline
P(a), Q(b) \Longrightarrow \exists x \exists y (P(x) \land Q(y)) \\ \hline
P(a), \exists x Q(x) \Longrightarrow \exists x \exists y (P(x) \land Q(y)) \\ \hline
\exists x P(x), \exists x Q(x) \Longrightarrow \exists x \exists y (P(x) \land Q(y)) \\ \hline
\exists x P(x) \land \exists x Q(x) \Longrightarrow \exists x \exists y (P(x) \land Q(y))
\end{array}
$$

(8-2) $\exists x \exists y (P(x) \land Q(y)) \Longrightarrow \exists x P(x) \land \exists x Q(x)$

$$
\begin{array}{c}
\dfrac{P(a) \Longrightarrow P(a)}{P(a) \Longrightarrow \exists x P(x)} \quad \dfrac{Q(b) \Longrightarrow Q(b)}{Q(b) \Longrightarrow \exists x Q(x)} \\ \hline
P(a), Q(b) \Longrightarrow \exists x P(x) \land \exists x Q(x) \\ \hline
P(a) \land Q(b) \Longrightarrow \exists x P(x) \land \exists x Q(x) \\ \hline
\exists y (P(a) \land Q(y)) \Longrightarrow \exists x P(x) \land \exists x Q(x) \\ \hline
\exists x \exists y (P(x) \land Q(y)) \Longrightarrow \exists x P(x) \land \exists x Q(x)
\end{array}
$$

(8'-2) $\exists x (P(x) \land Q(x)) \Longrightarrow \exists x P(x) \land \exists x Q(x)$ （逆は成り立たない）

$$
\begin{array}{c}
\dfrac{P(a) \Longrightarrow P(a)}{P(a) \Longrightarrow \exists x P(x)} \quad \dfrac{Q(a) \Longrightarrow Q(a)}{Q(a) \Longrightarrow \exists x Q(x)} \\ \hline
P(a), Q(a) \Longrightarrow \exists x P(x) \land \exists x Q(x) \\ \hline
P(a) \land Q(a) \Longrightarrow \exists x P(x) \land \exists x Q(x) \\ \hline
\exists x (P(x) \land Q(x)) \Longrightarrow \exists x P(x) \land \exists x Q(x)
\end{array}
$$

(9) $\forall x P(x) \to \forall x Q(x) \iff \exists x \forall y (P(x) \to Q(y))$ （ $\Longrightarrow$ は古典）

(9-1) $\forall x P(x) \to \forall x Q(x) \Longrightarrow \exists x \forall y (P(x) \to Q(y))$ （古典）

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Longrightarrow P(a)}{P(a) \Longrightarrow Q(b), P(a)}}{\Longrightarrow P(a) \to Q(b), P(a)}}{\Longrightarrow \forall y (P(a) \to Q(y)), P(a)}}{\Longrightarrow \exists x \forall y (P(x) \to Q(y)), P(a)}}{\Longrightarrow \exists x \forall y (P(x) \to Q(y)), \forall x P(x)}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q(c) \Longrightarrow Q(c)}{P(d), Q(c) \Longrightarrow Q(c)}}{Q(c) \Longrightarrow P(d) \to Q(c)}}{Q(c) \Longrightarrow \forall y (P(d) \to Q(y))}}{Q(c) \Longrightarrow \exists x \forall y (P(x) \to Q(y))}}{\forall x Q(x) \Longrightarrow \exists x \forall y (P(x) \to Q(y))}
}{\forall x P(x) \to \forall x Q(x) \Longrightarrow \exists x \forall y (P(x) \to Q(y))}
$$

(9-2) $\exists x \forall y (P(x) \to Q(y)) \Longrightarrow \forall x P(x) \to \forall x Q(x)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Longrightarrow P(a) \quad Q(b) \Longrightarrow Q(b)}{P(a), P(a) \to Q(b) \Longrightarrow Q(b)}}{P(a), \forall y (P(a) \to Q(y)) \Longrightarrow Q(b)}}{\forall x P(x), \forall y (P(a) \to Q(y)) \Longrightarrow Q(b)}}{\forall x P(x), \forall y (P(a) \to Q(y)) \Longrightarrow \forall x Q(x)}}{\forall x P(x), \exists x \forall y (P(x) \to Q(y)) \Longrightarrow \forall x Q(x)}}{\exists x \forall y (P(x) \to Q(y)) \Longrightarrow \forall x P(x) \to \forall x Q(x)}
$$

(10) $\exists x P(x) \to \exists x Q(x) \iff \exists y \forall x (P(x) \to Q(y))$ （ $\Longrightarrow$ は古典）

(10-1) $\exists x P(x) \to \exists x Q(x) \Longrightarrow \exists y \forall x (P(x) \to Q(y))$ (古典)

$$
\begin{array}{c}
\begin{array}{c}
P(b) \Longrightarrow P(b) \\ \hline
P(b) \Longrightarrow Q(a), P(b) \\ \hline
\Longrightarrow P(b) \to Q(a), P(b) \\ \hline
\Longrightarrow P(b) \to Q(a), \exists x P(x) \\ \hline
\Longrightarrow \forall x (P(x) \to Q(a)), \exists x P(x) \\ \hline
\Longrightarrow \exists y \forall x (P(x) \to Q(y)), \exists x P(x)
\end{array}
\quad
\begin{array}{c}
Q(c) \Longrightarrow Q(c) \\ \hline
P(d), Q(c) \Longrightarrow Q(c) \\ \hline
Q(c) \Longrightarrow P(d) \to Q(c) \\ \hline
Q(c) \Longrightarrow \forall x (P(x) \to Q(c)) \\ \hline
Q(c) \Longrightarrow \exists y \forall x (P(x) \to Q(y)) \\ \hline
\exists x Q(x) \Longrightarrow \exists y \forall x (P(x) \to Q(y))
\end{array}
\\ \hline
\exists x P(x) \to \exists x Q(x) \Longrightarrow \exists y \forall x (P(x) \to Q(y))
\end{array}
$$

(10-2) $\exists y \forall x (P(x) \to Q(y)) \Longrightarrow \exists x P(x) \to \exists x Q(x)$

$$
\begin{array}{c}
P(a) \Longrightarrow P(a) \quad Q(b) \Longrightarrow Q(b) \\ \hline
P(a), P(a) \to Q(b) \Longrightarrow Q(b) \\ \hline
P(a), P(a) \to Q(b) \Longrightarrow \exists x Q(x) \\ \hline
P(a), \forall x (P(x) \to Q(b)) \Longrightarrow \exists x Q(x) \\ \hline
P(a), \exists y \forall x (P(x) \to Q(y)) \Longrightarrow \exists x Q(x) \\ \hline
\exists x P(x), \exists y \forall x (P(x) \to Q(y)) \Longrightarrow \exists x Q(x) \\ \hline
\exists y \forall x (P(x) \to Q(y)) \Longrightarrow \exists x P(x) \to \exists x Q(x)
\end{array}
$$

(11) $\forall x P(x) \to \exists x Q(x) \iff \exists x(P(x) \to Q(x))$ （ $\Longrightarrow$ は古典）

(11-1) $\forall x P(x) \to \exists x Q(x) \Longrightarrow \exists x(P(x) \to Q(x))$ (古典)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Longrightarrow P(a)}{P(a) \Longrightarrow Q(a), P(a)}}{\Longrightarrow P(a) \to Q(a), P(a)}}{\Longrightarrow \exists x(P(x) \to Q(x)), P(a)}}{\Longrightarrow \exists x(P(x) \to Q(x)), \forall x P(x)} \quad \dfrac{\dfrac{\dfrac{\dfrac{Q(b) \Longrightarrow Q(b)}{Q(b), P(b) \Longrightarrow Q(b)}}{Q(b) \Longrightarrow P(b) \to Q(b)}}{Q(b) \Longrightarrow \exists x(P(x) \to Q(x))}}{\exists x Q(x) \Longrightarrow \exists x(P(x) \to Q(x))}}{\forall x P(x) \to \exists x Q(x) \Longrightarrow \exists x(P(x) \to Q(x))}$$

(11-2) $\exists x(P(x) \to Q(x)) \Longrightarrow \forall x P(x) \to \exists x Q(x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \Longrightarrow P(a) \quad Q(a) \Longrightarrow Q(a)}{P(a) \to Q(a), P(a) \Longrightarrow Q(a)}}{P(a) \to Q(a), P(a) \Longrightarrow \exists x Q(x)}}{P(a) \to Q(a), \forall x P(x) \Longrightarrow \exists x Q(x)}}{P(a) \to Q(a) \Longrightarrow \forall x P(x) \to \exists x Q(x)}}{\exists x(P(x) \to Q(x)) \Longrightarrow \forall x P(x) \to \exists x Q(x)}$$

(12) $\exists x P(x) \to \forall x Q(x) \iff \forall x \forall y (P(x) \to Q(y))$

(12') $\exists x P(x) \to \forall x Q(x) \implies \forall x (P(x) \to Q(x))$ （ $\impliedby$ は成り立たない）

(12-1) $\exists x P(x) \to \forall x Q(x) \implies \forall x \forall y (P(x) \to Q(y))$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(a) \implies P(a)}{P(a) \implies \exists x P(x)} \quad \cfrac{Q(b) \implies Q(b)}{\forall x Q(x) \implies Q(b)}}{\exists x P(x) \to \forall x Q(x), P(a) \implies Q(b)}}{\exists x P(x) \to \forall x Q(x) \implies P(a) \to Q(b)}}{\exists x P(x) \to \forall x Q(x) \implies \forall y (P(a) \to Q(y))}}{\exists x P(x) \to \forall x Q(x) \implies \forall x \forall y (P(x) \to Q(y))}}{}$$

(12-2) $\forall x \forall y (P(x) \to Q(y)) \implies \exists x P(x) \to \forall x Q(x)$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(a) \implies P(a) \quad Q(b) \implies Q(b)}{P(a) \to Q(b), P(a) \implies Q(b)}}{\forall y (P(a) \to Q(y)), P(a) \implies Q(b)}}{\forall x \forall y (P(x) \to Q(y)), P(a) \implies Q(b)}}{\forall x \forall y (P(x) \to Q(y)), P(a) \implies \forall x Q(x)}}{\forall x \forall y (P(x) \to Q(y)), \exists x P(x) \implies \forall x Q(x)}}{\forall x \forall y (P(x) \to Q(y)) \implies \exists x P(x) \to \forall x Q(x)}$$

(12'-1) $\exists x P(x) \to \forall x Q(x) \implies \forall x (P(x) \to Q(x))$

$$\cfrac{\cfrac{\cfrac{\cfrac{P(a) \implies P(a)}{P(a) \implies \exists x P(x)} \quad \cfrac{Q(a) \implies Q(a)}{\forall x Q(x) \implies Q(a)}}{\exists x P(x) \to \forall x Q(x), P(a) \implies Q(a)}}{\exists x P(x) \to \forall x Q(x) \implies P(a) \to Q(a)}}{\exists x P(x) \to \forall x Q(x) \implies \forall x (P(x) \to Q(x))}$$

(13) $\forall x \exists y (P(x) \to Q(y)) \iff \exists y \forall x (P(x) \to Q(y))$ （ $\Longrightarrow$ は古典）

(13') $\forall x \exists y P(x, y) \Longleftarrow \exists y \forall x P(x, y)$ （ $\Longrightarrow$ は成り立たない）

(13-1) $\forall x \exists y (P(x) \to Q(y)) \Longrightarrow \exists y \forall x (P(x) \to Q(y))$ （古典）

$$
\begin{array}{c}
\hfill Q(c) \Longrightarrow Q(c) \\ \hline
\hfill Q(c), P(d) \Longrightarrow Q(c) \\ \hline
\hfill Q(c) \Longrightarrow P(d) \to Q(c) \\ \hline
\hfill Q(c) \Longrightarrow \forall x (P(x) \to Q(c)) \\ \hline
P(b) \Longrightarrow P(b) \quad Q(c) \Longrightarrow \exists y \forall x (P(x) \to Q(y)) \\ \hline
P(b), P(b) \to Q(c) \Longrightarrow \exists y \forall x (P(x) \to Q(y)) \\ \hline
P(b), \exists y (P(b) \to Q(y)) \Longrightarrow \exists y \forall x (P(x) \to Q(y)) \\ \hline
P(b), \forall x \exists y (P(x) \to Q(y)) \Longrightarrow \exists y \forall x (P(x) \to Q(y)) \\ \hline
P(b), \forall x \exists y (P(x) \to Q(y)) \Longrightarrow \exists y \forall x (P(x) \to Q(y)), Q(a) \\ \hline
\forall x \exists y (P(x) \to Q(y)) \Longrightarrow \exists y \forall x (P(x) \to Q(y)), P(b) \to Q(a) \\ \hline
\forall x \exists y (P(x) \to Q(y)) \Longrightarrow \exists y \forall x (P(x) \to Q(y)), \forall x (P(x) \to Q(a)) \\ \hline
\forall x \exists y (P(x) \to Q(y)) \Longrightarrow \exists y \forall x (P(x) \to Q(y)), \exists y \forall x (P(x) \to Q(y)) \\ \hline
\forall x \exists y (P(x) \to Q(y)) \Longrightarrow \exists y \forall x (P(x) \to Q(y))
\end{array}
$$

(13-2) $\exists y \forall x (P(x) \to Q(y)) \Longrightarrow \forall x \exists y (P(x) \to Q(y))$

$$
\begin{array}{c}
P(a) \to Q(b) \Longrightarrow P(a) \to Q(b) \\ \hline
P(a) \to Q(b) \Longrightarrow \exists y (P(a) \to Q(y)) \\ \hline
\forall x (P(x) \to Q(b)) \Longrightarrow \exists y (P(a) \to Q(y)) \\ \hline
\exists y \forall x (P(x) \to Q(y)) \Longrightarrow \exists y (P(a) \to Q(y)) \\ \hline
\exists y \forall x (P(x) \to Q(y)) \Longrightarrow \forall x \exists y (P(x) \to Q(y))
\end{array}
$$

(13') $\exists y \forall x P(x, y) \Longrightarrow \forall x \exists y P(x, y)$ （逆は成り立たない）

$$
\begin{array}{c}
P(a, b) \Longrightarrow P(a, b) \\ \hline
P(a, b) \Longrightarrow \exists y P(a, y) \\ \hline
\forall x P(x, b) \Longrightarrow \exists y P(a, y) \\ \hline
\exists y \forall x P(x, y) \Longrightarrow \exists y P(a, y) \\ \hline
\exists y \forall x P(x, y) \Longrightarrow \forall x \exists y P(x, y)
\end{array}
$$