

1 形式演繹体系の証明例 (命題編)

1.1 問題

ド・モルガンの法則

- (1) $\neg P \wedge \neg Q \iff \neg(P \vee Q)$
- (2) $\neg P \vee \neg Q \iff \neg(P \wedge Q)$ (\iff は古典)

分配法則

- (3) $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$
- (4) $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$

吸収法則

- (5) $(P \vee Q) \wedge \neg Q \iff P \wedge \neg Q$
- (6) $(P \wedge Q) \vee \neg Q \iff P \vee \neg Q$ (\iff は古典)

対偶の性質

- (7) $P \rightarrow Q \iff \neg Q \rightarrow \neg P$ (\iff は古典)

\rightarrow の性質

- (8) $P \rightarrow Q \iff \neg P \vee Q$ (\iff は古典)
- (9) $\neg(P \rightarrow Q) \iff P \wedge \neg Q$ (\iff は古典)
- (10) $(P \rightarrow R) \wedge (Q \rightarrow R) \iff (P \vee Q) \rightarrow R$
- (11) $(P \rightarrow R) \vee (Q \rightarrow R) \iff (P \wedge Q) \rightarrow R$ (\iff は古典)
- (12) $(P \rightarrow Q) \wedge (\neg P \rightarrow R) \iff (P \wedge Q) \vee (\neg P \wedge R)$ (\iff は古典)

1.2 解答

ド・モルガンの法則

$$(1) \neg P \wedge \neg Q \iff \neg(P \vee Q)$$

$$(1-1) \neg P \wedge \neg Q \implies \neg(P \vee Q)$$

$$\frac{\frac{P \implies P \quad Q \implies Q}{P, \neg P \implies \quad Q, \neg Q \implies}}{P \vee Q, \neg P, \neg Q \implies} \\ \frac{\neg P, \neg Q \implies \neg(P \vee Q)}{\neg P \wedge \neg Q \implies \neg(P \vee Q)}$$

$$(1-2) \neg(P \vee Q) \implies \neg P \wedge \neg Q$$

$$\frac{\frac{P \implies P}{P \implies P \vee Q} \quad \frac{Q \implies Q}{Q \implies P \vee Q}}{P, \neg(P \vee Q) \implies \quad Q, \neg(P \vee Q) \implies} \\ \frac{\neg(P \vee Q) \implies \neg P \quad \neg(P \vee Q) \implies \neg Q}{\neg(P \vee Q) \implies \neg P \wedge \neg Q}$$

$$(2) \neg P \vee \neg Q \iff \neg(P \wedge Q) \quad (\Leftarrow \text{は古典})$$

$$(2-1) \neg P \vee \neg Q \implies \neg(P \wedge Q)$$

$$\frac{\frac{P \implies P \quad Q \implies Q}{P, \neg P \implies \quad Q, \neg Q \implies}}{P, Q, \neg P \vee \neg Q \implies} \\ \frac{P \wedge Q, \neg P \vee \neg Q \implies}{\neg P \vee \neg Q \implies \neg(P \wedge Q)}$$

$$(2-2) \neg(P \wedge Q) \implies \neg P \vee \neg Q \quad (\text{古典})$$

$$\frac{\frac{P \implies P \quad Q \implies Q}{P, Q \implies P \wedge Q}}{P, Q, \neg(P \wedge Q) \implies} \\ \frac{Q, \neg(P \wedge Q) \implies \neg P}{\neg(P \wedge Q) \implies \neg P, \neg Q} \\ \frac{\neg(P \wedge Q) \implies \neg P, \neg Q}{\neg(P \wedge Q) \implies \neg P \vee \neg Q}$$

分配法則

$$(3) P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$$

$$(3-1) P \vee (Q \wedge R) \implies (P \vee Q) \wedge (P \vee R)$$

$$\frac{\frac{P \implies P}{P \implies P \vee Q} \quad \frac{Q \implies Q}{Q \implies P \vee Q}}{P \vee (Q \wedge R) \implies P \vee Q} \quad \frac{\frac{P \implies P}{P \implies P \vee R} \quad \frac{R \implies R}{R \implies P \vee R}}{Q \wedge R \implies P \vee R}}{P \vee (Q \wedge R) \implies P \vee R}}{P \vee (Q \wedge R) \implies (P \vee Q) \wedge (P \vee R)}$$

$$(3-2) (P \vee Q) \wedge (P \vee R) \implies P \vee (Q \wedge R)$$

$$\frac{\frac{P \implies P}{P \implies P \vee (Q \wedge R)} \quad \frac{\frac{Q \implies Q}{Q, R \implies Q \wedge R} \quad \frac{R \implies R}{Q, R \implies P \vee (Q \wedge R)}}{Q, P \vee R \implies P \vee (Q \wedge R)}}{P \vee Q, P \vee R \implies P \vee (Q \wedge R)}}{(P \vee Q) \wedge (P \vee R) \implies P \vee (Q \wedge R)}$$

$$(4) P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$$

$$(4-1) P \wedge (Q \vee R) \implies (P \wedge Q) \vee (P \wedge R)$$

$$\frac{\frac{P \implies P}{P, Q \implies P \wedge Q} \quad \frac{Q \implies Q}{P, Q \implies P \wedge Q}}{P, Q \implies (P \wedge Q) \vee (P \wedge R)} \quad \frac{\frac{P \implies P}{P, R \implies P \wedge R} \quad \frac{R \implies R}{P, R \implies P \wedge R}}{P, R \implies (P \wedge Q) \vee (P \wedge R)}}{P, Q \vee R \implies (P \wedge Q) \vee (P \wedge R)}}{P \wedge (Q \vee R) \implies (P \wedge Q) \vee (P \wedge R)}$$

$$(4-2) (P \wedge Q) \vee (P \wedge R) \implies P \wedge (Q \vee R)$$

$$\frac{\frac{P \implies P}{P \wedge Q \implies P} \quad \frac{P \implies P}{P \wedge R \implies P}}{(P \wedge Q) \vee (P \wedge R) \implies P} \quad \frac{\frac{Q \implies Q}{Q \implies Q \vee R} \quad \frac{R \implies R}{R \implies Q \vee R}}{P \wedge Q \implies Q \vee R} \quad \frac{R \implies R}{P \wedge R \implies Q \vee R}}{(P \wedge Q) \vee (P \wedge R) \implies Q \vee R}}{(P \wedge Q) \vee (P \wedge R) \implies P \wedge (Q \vee R)}$$

吸収法則

(5) $(P \vee Q) \wedge \neg Q \iff P \wedge \neg Q$

(5-1) $(P \vee Q) \wedge \neg Q \implies P \wedge \neg Q$

$$\frac{\frac{P \implies P \quad \neg Q \implies \neg Q}{P, \neg Q \implies P \wedge \neg Q} \quad \frac{\frac{Q \implies Q}{Q, \neg Q \implies \text{}}{Q, \neg Q \implies P \wedge \neg Q}}{P \vee Q, \neg Q \implies P \wedge \neg Q}}{(P \vee Q) \wedge \neg Q \implies P \wedge \neg Q}$$

(5-2) $P \wedge \neg Q \implies (P \vee Q) \wedge \neg Q$

$$\frac{\frac{\frac{P \implies P}{P \implies P \vee Q}}{P \wedge \neg Q \implies P \vee Q} \quad \frac{\neg Q \implies \neg Q}{P \wedge \neg Q \implies \neg Q}}{P \wedge \neg Q \implies (P \vee Q) \wedge \neg Q}$$

(6) $(P \wedge Q) \vee \neg Q \iff P \vee \neg Q$ (\Leftarrow は古典)

(6-1) $(P \wedge Q) \vee \neg Q \implies P \vee \neg Q$

$$\frac{\frac{\frac{P \implies P}{P \implies P \vee \neg Q}}{P \wedge Q \implies P \vee \neg Q} \quad \frac{\neg Q \implies \neg Q}{\neg Q \implies P \vee \neg Q}}{(P \wedge Q) \vee \neg Q \implies P \vee \neg Q}$$

(6-2) $P \vee \neg Q \implies (P \wedge Q) \vee \neg Q$ (古典)

$$\frac{\frac{\frac{P \implies P \quad Q \implies Q}{P, Q \implies P \wedge Q}}{P \implies P \wedge Q, \neg Q} \quad \frac{\neg Q \implies \neg Q}{\neg Q \implies (P \wedge Q) \vee \neg Q}}{P \vee \neg Q \implies (P \wedge Q) \vee \neg Q}$$

対偶の性質

(7) $P \rightarrow Q \implies \neg Q \rightarrow \neg P$ (\Leftarrow は古典)

(7-1) $P \rightarrow Q \implies \neg Q \rightarrow \neg P$

$$\frac{\frac{\frac{P \implies P \quad Q \implies Q}{P, P \rightarrow Q \implies Q}}{P, \neg Q, P \rightarrow Q \implies}}{\neg Q, P \rightarrow Q \implies \neg P}}{P \rightarrow Q \implies \neg Q \rightarrow \neg P}$$

(7-2) $\neg Q \rightarrow \neg P \implies P \rightarrow Q$ (古典)

$$\frac{\frac{\frac{Q \implies Q \quad P \implies P}{\implies Q, \neg Q \quad P, \neg P \implies}}{P, \neg Q \rightarrow \neg P \implies Q}}{\neg Q \rightarrow \neg P \implies P \rightarrow Q}}$$

→ の性質

$$(8) P \rightarrow Q \iff \neg P \vee Q \quad (\implies \text{は古典})$$

$$(8-1) P \rightarrow Q \implies \neg P \vee Q \quad (\text{古典})$$

$$\frac{\frac{P \implies P \quad Q \implies Q}{P, P \rightarrow Q \implies Q}}{P \rightarrow Q \implies \neg P, Q}}{P \rightarrow Q \implies \neg P \vee Q}$$

$$(8-2) \neg P \vee Q \implies P \rightarrow Q$$

$$\frac{\frac{P \implies P}{P, \neg P \implies Q \implies Q}}{P, \neg P \vee Q \implies Q}}{\neg P \vee Q \implies P \rightarrow Q}$$

$$(9) \neg(P \rightarrow Q) \iff P \wedge \neg Q \quad (\implies \text{は古典})$$

$$(9-1) \neg(P \rightarrow Q) \implies P \wedge \neg Q \quad (\text{古典})$$

$$\frac{\frac{\frac{P \implies P}{P \implies P, Q}}{\implies P, P \rightarrow Q} \quad \frac{\frac{Q \implies Q}{P, Q \implies Q}}{Q \implies P \rightarrow Q}}{\frac{\neg(P \rightarrow Q) \implies P \quad \neg(P \rightarrow Q) \implies \neg Q}}{\neg(P \rightarrow Q) \implies P \wedge \neg Q}}$$

$$(9-2) P \wedge \neg Q \implies \neg(P \rightarrow Q)$$

$$\frac{\frac{P \implies P}{P \wedge \neg Q \implies P} \quad \frac{Q \implies Q}{Q, \neg Q \implies}}{P \rightarrow Q, P \wedge \neg Q \implies}}{P \wedge \neg Q \implies \neg(P \rightarrow Q)}$$

$$(10) (P \rightarrow R) \wedge (Q \rightarrow R) \iff (P \vee Q) \rightarrow R$$

$$(10-1) (P \rightarrow R) \wedge (Q \rightarrow R) \implies (P \vee Q) \rightarrow R$$

$$\frac{\frac{\frac{P \implies P \quad R \implies R}{P, P \rightarrow R \implies R} \quad \frac{Q \implies Q \quad R \implies R}{Q, Q \rightarrow R \implies R}}{P, (P \rightarrow R) \wedge (Q \rightarrow R) \implies R} \quad \frac{Q, (P \rightarrow R) \wedge (Q \rightarrow R) \implies R}}{P \vee Q, (P \rightarrow R) \wedge (Q \rightarrow R) \implies R}}{(P \rightarrow R) \wedge (Q \rightarrow R) \implies (P \vee Q) \rightarrow R}$$

$$(10-2) (P \vee Q) \rightarrow R \implies (P \rightarrow R) \wedge (Q \rightarrow R)$$

$$\frac{\frac{\frac{P \implies P}{P \implies P \vee Q} \quad R \implies R}{P, (P \vee Q) \rightarrow R \implies R} \quad \frac{Q \implies Q}{Q \implies P \vee Q} \quad R \implies R}{\frac{(P \vee Q) \rightarrow R \implies P \rightarrow R \quad (P \vee Q) \rightarrow R \implies Q \rightarrow R}}{(P \vee Q) \rightarrow R \implies (P \rightarrow R) \wedge (Q \rightarrow R)}}$$

(11) $(P \rightarrow R) \vee (Q \rightarrow R) \iff (P \wedge Q) \rightarrow R$ (\iff は古典)

(11-1) $(P \rightarrow R) \vee (Q \rightarrow R) \implies (P \wedge Q) \rightarrow R$

$$\frac{\frac{P \implies P \quad R \implies R}{P, P \rightarrow R \implies R} \quad \frac{Q \implies Q \quad R \implies R}{Q, Q \rightarrow R \implies R}}{\frac{P \wedge Q, P \rightarrow R \implies R \quad P \wedge Q, Q \rightarrow R \implies R}{P \wedge Q, (P \rightarrow R) \vee (Q \rightarrow R) \implies R}} \implies R$$

$$\frac{P \wedge Q, (P \rightarrow R) \vee (Q \rightarrow R) \implies R}{(P \rightarrow R) \vee (Q \rightarrow R) \implies (P \wedge Q) \rightarrow R}$$

(11-2) $(P \wedge Q) \rightarrow R \implies (P \rightarrow R) \vee (Q \rightarrow R)$ (古典)

$$\frac{\frac{P \implies P \quad Q \implies Q}{P, Q \implies P \wedge Q} \quad R \implies R}{\frac{P, Q, (P \wedge Q) \rightarrow R \implies R}{P, Q, (P \wedge Q) \rightarrow R \implies R, R}} \implies R$$

$$\frac{P, Q, (P \wedge Q) \rightarrow R \implies R, R}{Q, (P \wedge Q) \rightarrow R \implies P \rightarrow R, R} \implies R$$

$$\frac{Q, (P \wedge Q) \rightarrow R \implies P \rightarrow R, R}{(P \wedge Q) \rightarrow R \implies P \rightarrow R, Q \rightarrow R} \implies R$$

$$\frac{(P \wedge Q) \rightarrow R \implies P \rightarrow R, Q \rightarrow R}{(P \wedge Q) \rightarrow R \implies (P \rightarrow R) \vee (Q \rightarrow R)}$$

(12) $(P \rightarrow Q) \wedge (\neg P \rightarrow R) \iff (P \wedge Q) \vee (\neg P \wedge R)$ (\implies は古典)

(12-1) $(P \rightarrow Q) \wedge (\neg P \rightarrow R) \implies (P \wedge Q) \vee (\neg P \wedge R)$ (古典)

$$\frac{\frac{P \implies P}{\implies P, \neg P} \quad \frac{P \implies P}{\implies P, \neg P} \quad \frac{R \implies R}{\neg P \rightarrow R \implies P, R}}{\frac{\neg P \rightarrow R \implies P, \neg P \wedge R}{\neg P \rightarrow R \implies P, \neg P \wedge R}} \quad \frac{\text{同左}}{\frac{\neg P \rightarrow R \implies P, \neg P \wedge R \quad Q \implies Q}{P \rightarrow Q, \neg P \rightarrow R \implies Q, \neg P \wedge R}}$$

$$\frac{P \rightarrow Q, \neg P \rightarrow R \implies P \wedge Q, \neg P \wedge R}{(P \rightarrow Q) \wedge (\neg P \rightarrow R) \implies P \wedge Q, \neg P \wedge R}$$

$$\frac{(P \rightarrow Q) \wedge (\neg P \rightarrow R) \implies P \wedge Q, \neg P \wedge R}{(P \rightarrow Q) \wedge (\neg P \rightarrow R) \implies (P \wedge Q) \vee (\neg P \wedge R)}$$

(12-2) $(P \wedge Q) \vee (\neg P \wedge R) \implies (P \rightarrow Q) \wedge (\neg P \rightarrow R)$

$$\frac{\frac{Q \implies Q}{P \wedge Q \implies Q} \quad \frac{P \implies P}{P, \neg P \implies} \quad \frac{P \implies P}{\neg P, P \implies} \quad \frac{R \implies R}{\neg P \wedge R \implies R}}{\frac{P, (P \wedge Q) \vee (\neg P \wedge R) \implies Q}{P, (P \wedge Q) \vee (\neg P \wedge R) \implies Q} \quad \frac{\neg P, (P \wedge Q) \vee (\neg P \wedge R) \implies R}{\neg P, (P \wedge Q) \vee (\neg P \wedge R) \implies R}}$$

$$\frac{P, (P \wedge Q) \vee (\neg P \wedge R) \implies Q \quad \neg P, (P \wedge Q) \vee (\neg P \wedge R) \implies R}{(P \wedge Q) \vee (\neg P \wedge R) \implies P \rightarrow Q \quad (P \wedge Q) \vee (\neg P \wedge R) \implies \neg P \rightarrow R}$$

$$\frac{(P \wedge Q) \vee (\neg P \wedge R) \implies P \rightarrow Q \quad (P \wedge Q) \vee (\neg P \wedge R) \implies \neg P \rightarrow R}{(P \wedge Q) \vee (\neg P \wedge R) \implies (P \rightarrow Q) \wedge (\neg P \rightarrow R)}$$