問題 $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x^3}$ の値を求めなさい

解答

$$0 \geqq \cos x - 1 \quad (x>0) \tag{1}$$

$$\therefore\ 0 \geqq \int_0^x (\cos t - 1)\,dt = \sin x - x \quad (x>0) \tag{2}$$

$$\therefore\ 0 \geqq \int_0^x (\sin t - t)\,dt = -\cos x + 1 - \frac{x^2}{2} \quad (x>0) \tag{3}$$

$$\therefore\ 0 \geqq \int_0^x \left(-\cos t + 1 - \frac{t^2}{2}\right)dt = -\sin x + x - \frac{x^3}{6} \quad (x>0) \tag{4}$$

$$\therefore\ 0 \geqq \int_0^x \left(-\sin t + t - \frac{t^3}{6}\right)dt = \cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24} \quad (x>0) \tag{5}$$

$$\therefore\ 0 \geqq \int_0^x \left(\cos t - 1 + \frac{t^2}{2} - \frac{t^4}{24}\right)dt = \sin x - x + \frac{x^3}{6} - \frac{x^5}{120} \quad (x>0) \tag{6}$$

(4),(6) より

$$\frac{x^3}{6} \geqq x - \sin x \geqq \frac{x^3}{6} - \frac{x^5}{120} \quad (x>0) \tag{7}$$

$$\therefore\ \frac{1}{6} \geqq \frac{x-\sin x}{x^3} \geqq \frac{1}{6} - \frac{x^2}{120} \quad (x>0) \tag{8}$$

$$\therefore\ \lim_{x\to +0}\frac{x-\sin x}{x^3} = \frac{1}{6} \tag{9}$$

$x<0$ について $x=-t$ とおくと

$$\frac{x-\sin x}{x^3} = \frac{-t-\sin(-t)}{(-t)^3} = \frac{t-\sin t}{t^3} \tag{10}$$

$$\therefore\ \lim_{x\to -0}\frac{x-\sin x}{x^3} = \lim_{t\to +0}\frac{t-\sin t}{t^3} = \frac{1}{6} \tag{11}$$

したがって

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{1}{6} \tag{12}$$

参考

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$