

2 自然演繹体系の証明例 (述語編)

2.1 問題

$$(1) \neg \exists x P(x) \iff \forall x \neg P(x)$$

$$(2) \neg \forall x P(x) \iff \exists x \neg P(x) \quad (\implies \text{は古典})$$

$$(3) \forall x \forall y P(x, y) \iff \forall y \forall x P(x, y)$$

$$(4) \exists x \exists y P(x, y) \iff \exists y \exists x P(x, y)$$

$$(5) \forall x P(x) \wedge \forall x Q(x) \iff \forall x (P(x) \wedge Q(x))$$

$$(6) \exists x P(x) \vee \exists x Q(x) \iff \exists x (P(x) \vee Q(x))$$

$$(7) \forall x P(x) \vee \forall x Q(x) \iff \forall x \forall y (P(x) \vee Q(y)) \quad (\iff \text{は古典})$$

$$(7') \forall x P(x) \vee \forall x Q(x) \implies \forall x (P(x) \vee Q(x)) \quad (\iff \text{は成り立たない})$$

$$(8) \exists x P(x) \wedge \exists x Q(x) \iff \exists x \exists y (P(x) \wedge Q(y))$$

$$(8') \exists x P(x) \wedge \exists x Q(x) \iff \exists x (P(x) \wedge Q(x)) \quad (\implies \text{は成り立たない})$$

$$(9) \forall x P(x) \rightarrow \forall x Q(x) \iff \exists x \forall y (P(x) \rightarrow Q(y)) \quad (\implies \text{は古典})$$

$$(10) \exists x P(x) \rightarrow \exists x Q(x) \iff \exists y \forall x (P(x) \rightarrow Q(y)) \quad (\implies \text{は古典})$$

$$(11) \forall x P(x) \rightarrow \exists x Q(x) \iff \exists x (P(x) \rightarrow Q(x)) \quad (\implies \text{は古典})$$

$$(12) \exists x P(x) \rightarrow \forall x Q(x) \iff \forall x \forall y (P(x) \rightarrow Q(y))$$

$$(12') \exists x P(x) \rightarrow \forall x Q(x) \implies \forall x (P(x) \rightarrow Q(x)) \quad (\iff \text{は成り立たない})$$

$$(13) \forall x \exists y (P(x) \rightarrow Q(y)) \iff \exists y \forall x (P(x) \rightarrow Q(y)) \quad (\implies \text{は古典})$$

$$(13') \forall x \exists y P(x, y) \iff \exists y \forall x P(x, y) \quad (\implies \text{は成り立たない})$$

2.2 解答

$$(1) \quad \neg\exists xP(x) \iff \forall x\neg P(x)$$

$$(1-1) \quad \neg\exists xP(x) \implies \forall x\neg P(x)$$

$$\frac{\frac{\text{仮定}^{(*1)} \quad P(a)}{\exists xP(x)} \quad \text{前提} \quad \neg\exists xP(x)}{\frac{\perp}{\neg P(a)} \quad (*1)} \quad \forall x\neg P(x)$$

$$(1-2) \quad \forall x\neg P(x) \implies \neg\exists xP(x)$$

$$\frac{\frac{\text{仮定}^{(*1)} \quad \exists xP(x)}{\frac{\text{仮定}^{(*2)} \quad \forall x\neg P(x)}{\neg P(a)} \quad \text{前提}}{\perp} \quad (*2)}{\frac{\perp}{\neg\exists xP(x)} \quad (*1)}$$

$$(2) \quad \neg\forall xP(x) \iff \exists x\neg P(x) \quad (\implies \text{は古典})$$

$$(2-1) \quad \neg\forall xP(x) \implies \exists x\neg P(x) \quad (\text{古典})$$

$$\frac{\frac{\text{仮定}^{(*1)} \quad \neg P(a)}{\exists x\neg P(x)} \quad \text{仮定}^{(*2)} \quad \neg\exists x\neg P(x)}{\frac{\perp}{\neg\neg P(a)} \quad (*1)} \quad \frac{\frac{P(a)}{\forall xP(x)} \quad \text{前提} \quad \neg\forall xP(x)}{\frac{\perp}{\neg\neg\exists x\neg P(x)} \quad (*2)} \quad \exists x\neg P(x)$$

$$(2-2) \quad \exists x\neg P(x) \implies \neg\forall xP(x)$$

$$\frac{\frac{\text{前提} \quad \exists x\neg P(x)}{\frac{\text{仮定}^{(*2)} \quad \forall xP(x)}{P(a)} \quad \text{仮定}^{(*1)} \quad \neg P(a)}{\perp} \quad (*1)}{\frac{\perp}{\neg\forall xP(x)} \quad (*2)}$$

$$(5) \quad \forall xP(x) \wedge \forall xQ(x) \iff \forall x(P(x) \wedge Q(x))$$

$$(5-1) \quad \forall xP(x) \wedge \forall xQ(x) \implies \forall x(P(x) \wedge Q(x))$$

$$\frac{\frac{\text{前提} \quad \forall xP(x) \wedge \forall xQ(x)}{\forall xP(x)} \quad \frac{\text{前提} \quad \forall xP(x) \wedge \forall xQ(x)}{\forall xQ(x)}}{\frac{P(a) \quad Q(a)}{P(a) \wedge Q(a)}} \quad \frac{P(a) \wedge Q(a)}{\forall x(P(x) \wedge Q(x))}$$

$$(5-2) \quad \forall x(P(x) \wedge Q(x)) \implies \forall xP(x) \wedge \forall xQ(x)$$

$$\frac{\frac{\text{前提} \quad \forall x(P(x) \wedge Q(x))}{P(a) \wedge Q(a)} \quad \frac{\text{前提} \quad \forall x(P(x) \wedge Q(x))}{P(a) \wedge Q(a)}}{\frac{P(a) \quad Q(a)}{\forall xP(x)} \quad \frac{P(a) \quad Q(a)}{\forall xQ(x)}} \quad \frac{\forall xP(x) \quad \forall xQ(x)}{\forall xP(x) \wedge \forall xQ(x)}$$

$$(6) \quad \exists xP(x) \vee \exists xQ(x) \iff \exists x(P(x) \vee Q(x))$$

$$(6-1) \quad \exists xP(x) \vee \exists xQ(x) \implies \exists x(P(x) \vee Q(x))$$

$$\frac{\text{前提} \quad \exists xP(x) \vee \exists xQ(x) \quad \frac{\text{假定}^{(*1)} \quad P(a)}{P(a) \vee Q(a)} \quad \frac{\text{假定}^{(*2)} \quad Q(a)}{P(a) \vee Q(a)}}{\frac{\text{假定}^{(*3)} \quad \exists xP(x) \quad \frac{P(a) \vee Q(a)}{\exists x(P(x) \vee Q(x))}}{\exists x(P(x) \vee Q(x))} \quad \frac{\text{假定}^{(*3)} \quad \exists xQ(x) \quad \frac{P(a) \vee Q(a)}{\exists x(P(x) \vee Q(x))}}{\exists x(P(x) \vee Q(x))}}{\exists x(P(x) \vee Q(x))} \quad (*2) \quad (*3)$$

$$(6-2) \quad \exists x(P(x) \vee Q(x)) \implies \exists xP(x) \vee \exists xQ(x)$$

$$\frac{\text{前提} \quad \exists x(P(x) \vee Q(x)) \quad \frac{\text{假定}^{(*1)} \quad P(a)}{\exists xP(x)} \quad \frac{\text{假定}^{(*1)} \quad Q(a)}{\exists xQ(x)}}{\frac{\text{假定}^{(*2)} \quad P(a) \vee Q(a) \quad \frac{\exists xP(x) \quad \exists xQ(x)}{\exists xP(x) \vee \exists xQ(x)}}{\exists xP(x) \vee \exists xQ(x)} \quad \frac{\exists xP(x) \vee \exists xQ(x)}{\exists xP(x) \vee \exists xQ(x)}}{\exists xP(x) \vee \exists xQ(x)} \quad (*1) \quad (*2)$$

(7) $\forall xP(x) \vee \forall xQ(x) \iff \forall x\forall y(P(x) \vee Q(y))$ (\Leftarrow は古典)

(7') $\forall xP(x) \vee \forall xQ(x) \implies \forall x(P(x) \vee Q(x))$ (\Leftarrow は成り立たない)

(7-1) $\forall xP(x) \vee \forall xQ(x) \implies \forall x\forall y(P(x) \vee Q(y))$

$$\frac{\text{前提} \quad \frac{\forall xP(x)}{P(a)} \quad \frac{\forall xQ(x)}{Q(b)}}{\frac{P(a) \vee Q(b)}{P(a) \vee Q(b)}} \quad (*1)$$

$$\frac{\forall y(P(a) \vee Q(y))}{\forall x\forall y(P(x) \vee Q(y))}$$

(7-2) $\forall x\forall y(P(x) \vee Q(y)) \implies \forall xP(x) \vee \forall xQ(x)$ (古典)

$$\frac{\text{前提} \quad \frac{\forall x\forall y(P(x) \vee Q(y))}{\forall y(P(a) \vee Q(y))} \quad \frac{\text{仮定}^{(*1)} \quad \text{仮定}^{(*2)}}{P(a) \quad \neg P(a)}}{\frac{P(a) \vee Q(b)}{P(a) \vee Q(b)}} \quad (*1)$$

$$\frac{Q(b)}{\forall xQ(x)} \quad \frac{\text{仮定}^{(*1)}}{Q(b)} \quad \frac{\text{仮定}^{(*3)}}{\neg(\forall xP(x) \vee \forall xQ(x))}$$

$$\frac{\perp}{\neg\neg P(a)} \quad (*2)$$

$$\frac{P(a)}{\forall xP(x)} \quad \frac{\text{仮定}^{(*3)}}{\neg(\forall xP(x) \vee \forall xQ(x))}$$

$$\frac{\perp}{\neg\neg\forall xP(x) \vee \forall xQ(x)} \quad (*3)$$

$$\frac{\perp}{\forall xP(x) \vee \forall xQ(x)}$$

(7'-1) $\forall xP(x) \vee \forall xQ(x) \implies \forall x(P(x) \vee Q(x))$ (逆は成り立たない)

$$\frac{\text{前提} \quad \frac{\forall xP(x)}{P(a)} \quad \frac{\forall xQ(x)}{Q(a)}}{\frac{P(a) \vee Q(a)}{P(a) \vee Q(a)}} \quad (*1)$$

$$\frac{P(a) \vee Q(a)}{\forall x(P(x) \vee Q(x))}$$

$$(8) \quad \exists xP(x) \wedge \exists xQ(x) \iff \exists x\exists y(P(x) \wedge Q(y))$$

$$(8') \quad \exists xP(x) \wedge \exists xQ(x) \iff \exists x(P(x) \wedge Q(x)) \quad (\implies \text{は成り立たない})$$

$$(8-1) \quad \exists xP(x) \wedge \exists xQ(x) \implies \exists x\exists y(P(x) \wedge Q(y))$$

$$\begin{array}{c} \text{前提} \\ \frac{\exists xP(x) \wedge \exists xQ(x)}{\exists xP(x)} \\ \hline \frac{\frac{\frac{\frac{\text{仮定}^{(*2)} \quad \text{仮定}^{(*1)}}{P(a) \quad Q(b)}}{P(a) \wedge Q(b)}}{\exists y(P(a) \wedge Q(y))}}{\exists x\exists y(P(x) \wedge Q(y))} \text{前提}}{\exists x\exists y(P(x) \wedge Q(y))} \text{前提} \\ \hline \frac{\frac{\exists xQ(x)}{\exists x\exists y(P(x) \wedge Q(y))} \text{前提}}{\exists x\exists y(P(x) \wedge Q(y))} \text{前提} \end{array}$$

$$(8-2) \quad \exists x\exists y(P(x) \wedge Q(y)) \implies \exists xP(x) \wedge \exists xQ(x)$$

$$\begin{array}{c} \text{前提} \\ \frac{\exists x\exists y(P(x) \wedge Q(y))}{\exists xP(x) \wedge \exists xQ(x)} \text{前提} \\ \hline \frac{\frac{\frac{\frac{\text{仮定}^{(*1)} \quad \text{仮定}^{(*1)}}{P(a) \wedge Q(b) \quad P(a) \wedge Q(b)}}{\frac{P(a)}{\exists xP(x)} \quad \frac{Q(b)}{\exists xQ(x)}}}{\exists xP(x) \wedge \exists xQ(x)} \text{前提}}{\exists xP(x) \wedge \exists xQ(x)} \text{前提} \end{array}$$

$$(8'-2) \quad \exists x(P(x) \wedge Q(x)) \implies \exists xP(x) \wedge \exists xQ(x) \quad (\text{逆は成り立たない})$$

$$\begin{array}{c} \text{前提} \\ \frac{\exists x(P(x) \wedge Q(x))}{\exists xP(x) \wedge \exists xQ(x)} \text{前提} \\ \hline \frac{\frac{\frac{\text{仮定}^{(*1)} \quad \text{仮定}^{(*1)}}{P(a) \wedge Q(a) \quad P(a) \wedge Q(a)}}{\frac{P(a)}{\exists xP(x)} \quad \frac{Q(a)}{\exists xQ(x)}}}{\exists xP(x) \wedge \exists xQ(x)} \text{前提} \end{array}$$

(9) $\forall xP(x) \rightarrow \forall xQ(x) \iff \exists x\forall y(P(x) \rightarrow Q(y))$ (\implies は古典)

(9-1) $\forall xP(x) \rightarrow \forall xQ(x) \implies \exists x\forall y(P(x) \rightarrow Q(y))$ (古典)

$$\begin{array}{c}
 \text{仮定}^{(*1)} \quad \text{仮定}^{(*2)} \\
 \frac{P(a) \quad \neg P(a)}{\perp} \\
 \frac{\perp}{Q(b)} \\
 \frac{Q(b)}{P(a) \rightarrow Q(b)} \quad (*1) \\
 \frac{P(a) \rightarrow Q(b)}{\forall y(P(a) \rightarrow Q(y))} \\
 \frac{\forall y(P(a) \rightarrow Q(y))}{\exists x\forall y(P(x) \rightarrow Q(y))} \quad \text{仮定}^{(*3)} \quad \neg\exists x\forall y(P(x) \rightarrow Q(y)) \\
 \hline
 \frac{\perp}{\neg\neg P(a)} \quad (*2) \\
 \frac{\neg\neg P(a)}{P(a)} \\
 \frac{P(a)}{\forall xP(x)} \quad \text{前提} \quad \forall xP(x) \rightarrow \forall xQ(x) \\
 \hline
 \frac{\forall xQ(x)}{Q(d)} \\
 \frac{Q(d)}{P(c) \rightarrow Q(d)} \\
 \frac{P(c) \rightarrow Q(d)}{\forall y(P(c) \rightarrow Q(y))} \\
 \frac{\forall y(P(c) \rightarrow Q(y))}{\exists x\forall y(P(x) \rightarrow Q(y))} \quad \text{仮定}^{(*3)} \quad \neg\exists x\forall y(P(x) \rightarrow Q(y)) \\
 \hline
 \frac{\perp}{\neg\neg\exists x\forall y(P(x) \rightarrow Q(y))} \quad (*3) \\
 \frac{\neg\neg\exists x\forall y(P(x) \rightarrow Q(y))}{\exists x\forall y(P(x) \rightarrow Q(y))}
 \end{array}$$

(10) $\exists xP(x) \rightarrow \exists xQ(x) \iff \exists y\forall x(P(x) \rightarrow Q(y))$ (\implies は古典)

(10-1) $\exists xP(x) \rightarrow \exists xQ(x) \implies \exists y\forall x(P(x) \rightarrow Q(y))$ (古典)

$$\begin{array}{c}
 \text{仮定}^{(*1)} \\
 \frac{P(a)}{\exists xP(x)} \quad \text{仮定}^{(*2)} \\
 \frac{}{\neg\exists xP(x)} \\
 \frac{\perp}{Q(b)} \\
 \frac{}{P(a) \rightarrow Q(b)} \quad (*1) \\
 \frac{\forall x(P(x) \rightarrow Q(b))}{\exists y\forall x(P(x) \rightarrow Q(y))} \quad \text{仮定}^{(*4)} \\
 \frac{}{\neg\exists y\forall x(P(x) \rightarrow Q(y))} \\
 \frac{\perp}{\neg\neg\exists xP(x)} \quad (*2) \\
 \frac{}{\exists xP(x)} \\
 \frac{}{\exists xQ(x)} \quad \text{前提} \\
 \frac{}{\exists xP(x) \rightarrow \exists xQ(x)} \\
 \frac{}{Q(d)} \quad \text{仮定}^{(*3)} \\
 \frac{}{P(c) \rightarrow Q(d)} \\
 \frac{\forall x(P(x) \rightarrow Q(d))}{\exists y\forall x(P(x) \rightarrow Q(y))} \quad \text{仮定}^{(*4)} \\
 \frac{}{\neg\exists y\forall x(P(x) \rightarrow Q(y))} \\
 \frac{}{\exists xQ(x)} \quad \perp \quad (*3) \\
 \frac{}{\neg\neg\exists y\forall x(P(x) \rightarrow Q(y))} \quad (*4) \\
 \frac{}{\exists y\forall x(P(x) \rightarrow Q(y))}
 \end{array}$$

(10-2) $\exists y\forall x(P(x) \rightarrow Q(y)) \implies \exists xP(x) \rightarrow \exists xQ(x)$

$$\begin{array}{c}
 \text{仮定}^{(*3)} \\
 \frac{\forall x(P(x) \rightarrow Q(b))}{P(a) \rightarrow Q(b)} \\
 \frac{}{P(a)} \\
 \frac{}{Q(b)} \\
 \frac{}{\exists xP(x)} \quad \text{仮定}^{(*2)} \\
 \frac{}{\exists xQ(x)} \\
 \frac{}{\exists xQ(x)} \quad (*1) \\
 \frac{}{\exists xP(x) \rightarrow \exists xQ(x)} \quad (*2) \\
 \frac{}{\exists y\forall x(P(x) \rightarrow Q(y))} \quad \text{前提} \\
 \frac{}{\exists xP(x) \rightarrow \exists xQ(x)} \quad (*3)
 \end{array}$$

(11) $\forall xP(x) \rightarrow \exists xQ(x) \iff \exists x(P(x) \rightarrow Q(x))$ (\implies は古典)

(11-1) $\forall xP(x) \rightarrow \exists xQ(x) \implies \exists x(P(x) \rightarrow Q(x))$ (古典)

$$\begin{array}{c}
 \text{仮定}^{(*1)} \quad \text{仮定}^{(*2)} \\
 \frac{P(a) \quad \neg P(a)}{\perp} \\
 \frac{\perp}{Q(a)} \\
 \frac{Q(a)}{P(a) \rightarrow Q(a)} \quad (*1) \\
 \frac{P(a) \rightarrow Q(a)}{\exists x(P(x) \rightarrow Q(x))} \quad \text{仮定}^{(*4)} \\
 \frac{\exists x(P(x) \rightarrow Q(x)) \quad \neg \exists x(P(x) \rightarrow Q(x))}{\perp} \quad (*2) \\
 \frac{\perp}{\neg \neg P(a)} \quad (*2) \\
 \frac{\neg \neg P(a)}{P(a)} \\
 \frac{P(a)}{\forall xP(x)} \\
 \text{前提} \quad \frac{\forall xP(x) \rightarrow \exists xQ(x)}{\exists xQ(x)} \quad \frac{\text{仮定}^{(*3)} \quad Q(b)}{P(b) \rightarrow Q(b)} \quad \frac{P(b) \rightarrow Q(b)}{\exists x(P(x) \rightarrow Q(x))} \quad \frac{\text{仮定}^{(*4)} \quad \neg \exists x(P(x) \rightarrow Q(x))}{\neg \exists x(P(x) \rightarrow Q(x))} \\
 \frac{\exists xQ(x) \quad \exists x(P(x) \rightarrow Q(x)) \quad \neg \exists x(P(x) \rightarrow Q(x))}{\perp} \quad (*3) \\
 \frac{\perp}{\exists x(P(x) \rightarrow Q(x))} \quad (*4) \\
 \frac{\exists x(P(x) \rightarrow Q(x))}{\exists x(P(x) \rightarrow Q(x))}
 \end{array}$$

(11-2) $\exists x(P(x) \rightarrow Q(x)) \implies \forall xP(x) \rightarrow \exists xQ(x)$

$$\begin{array}{c}
 \text{仮定}^{(*2)} \\
 \frac{\forall xP(x)}{P(a)} \quad \text{仮定}^{(*1)} \\
 \frac{P(a) \quad P(a) \rightarrow Q(a)}{Q(a)} \\
 \frac{Q(a)}{\exists xQ(x)} \quad (*1) \\
 \text{前提} \quad \frac{\exists x(P(x) \rightarrow Q(x)) \quad \exists xQ(x)}{\exists xQ(x)} \\
 \frac{\exists xQ(x)}{\forall xP(x) \rightarrow \exists xQ(x)} \quad (*2)
 \end{array}$$

$$(12) \quad \exists xP(x) \rightarrow \forall xQ(x) \iff \forall x\forall y(P(x) \rightarrow Q(y))$$

$$(12') \quad \exists xP(x) \rightarrow \forall xQ(x) \implies \forall x(P(x) \rightarrow Q(x)) \quad (\iff \text{は成り立たない})$$

$$(12-1) \quad \exists xP(x) \rightarrow \forall xQ(x) \implies \forall x\forall y(P(x) \rightarrow Q(y))$$

$$\begin{array}{c} \text{仮定}^{(*1)} \\ \frac{P(a)}{\exists xP(x)} \quad \text{前提} \\ \frac{\exists xP(x) \rightarrow \forall xQ(x)}{\forall xQ(x)} \\ \frac{Q(b)}{P(a) \rightarrow Q(b)} \quad (*1) \\ \frac{\forall y(P(a) \rightarrow Q(y))}{\forall x\forall y(P(x) \rightarrow Q(y))} \end{array}$$

$$(12-2) \quad \forall x\forall y(P(x) \rightarrow Q(y)) \implies \exists xP(x) \rightarrow \forall xQ(x)$$

$$\begin{array}{c} \text{前提} \\ \frac{\forall x\forall y(P(x) \rightarrow Q(y))}{\forall y(P(a) \rightarrow Q(y))} \\ \frac{\text{仮定}^{(*1)} \quad P(a)}{P(a) \rightarrow Q(b)} \\ \frac{\exists xP(x) \quad Q(b)}{Q(b)} \quad (*1) \\ \frac{Q(b)}{\forall xQ(x)} \\ \frac{\forall xQ(x)}{\exists xP(x) \rightarrow \forall xQ(x)} \quad (*2) \end{array}$$

$$(12'-1) \quad \exists xP(x) \rightarrow \forall xQ(x) \implies \forall x(P(x) \rightarrow Q(x))$$

$$\begin{array}{c} \text{仮定}^{(*1)} \\ \frac{P(a)}{\exists xP(x)} \quad \text{前提} \\ \frac{\exists xP(x) \rightarrow \forall xQ(x)}{\forall xQ(x)} \\ \frac{Q(a)}{P(a) \rightarrow Q(a)} \quad (*1) \\ \frac{P(a) \rightarrow Q(a)}{\forall x(P(x) \rightarrow Q(x))} \end{array}$$

(13) $\forall x \exists y (P(x) \rightarrow Q(y)) \iff \exists y \forall x (P(x) \rightarrow Q(y))$ (\implies は古典)

(13') $\forall x \exists y P(x, y) \iff \exists y \forall x P(x, y)$ (\implies は成り立たない)

(13-1) $\forall x \exists y (P(x) \rightarrow Q(y)) \implies \exists y \forall x (P(x) \rightarrow Q(y))$ (古典)

$$\begin{array}{c}
 \begin{array}{c}
 \text{仮定}^{(*2)} \quad \text{仮定}^{(*1)} \\
 \frac{P(a) \quad P(a) \rightarrow Q(b)}{Q(b)} \\
 \frac{P(a) \rightarrow Q(b)}{\forall x (P(x) \rightarrow Q(b))} \\
 \frac{\forall x (P(x) \rightarrow Q(b))}{\exists y \forall x (P(x) \rightarrow Q(y))} \quad \text{仮定}^{(*3)} \\
 \frac{\exists y \forall x (P(x) \rightarrow Q(y)) \quad \neg \exists y \forall x (P(x) \rightarrow Q(y))}{\perp} \\
 \frac{\exists y (P(a) \rightarrow Q(y)) \quad \perp}{Q(d)} \quad (*1) \\
 \frac{Q(d)}{P(a) \rightarrow Q(d)} \quad (*2) \\
 \frac{\forall x (P(x) \rightarrow Q(d))}{\exists y \forall x (P(x) \rightarrow Q(y))} \\
 \frac{\exists y \forall x (P(x) \rightarrow Q(y)) \quad \neg \exists y \forall x (P(x) \rightarrow Q(y))}{\perp} \quad \text{仮定}^{(*3)} \\
 \frac{\perp}{\exists y \forall x (P(x) \rightarrow Q(y))} \quad (*3)
 \end{array}
 \end{array}$$

(13-2) $\exists y \forall x (P(x) \rightarrow Q(y)) \implies \forall x \exists y (P(x) \rightarrow Q(y))$

$$\begin{array}{c}
 \begin{array}{c}
 \text{仮定}^{(*1)} \\
 \frac{\forall x (P(x) \rightarrow Q(b))}{P(a) \rightarrow Q(b)} \\
 \frac{P(a) \rightarrow Q(b)}{\exists y (P(a) \rightarrow Q(y))} \\
 \frac{\exists y \forall x (P(x) \rightarrow Q(y)) \quad \forall x \exists y (P(x) \rightarrow Q(y))}{\forall x \exists y (P(x) \rightarrow Q(y))} \quad (*1)
 \end{array}
 \end{array}$$

(13') $\exists y \forall x P(x, y) \implies \forall x \exists y P(x, y)$ (逆は成り立たない)

$$\begin{array}{c}
 \begin{array}{c}
 \text{仮定}^{(*1)} \\
 \frac{\forall x P(x, b)}{P(a, b)} \\
 \frac{P(a, b)}{\exists y P(a, y)} \\
 \frac{\exists y \forall x P(x, y) \quad \forall x \exists y P(x, y)}{\forall x \exists y P(x, y)} \quad (*1)
 \end{array}
 \end{array}$$