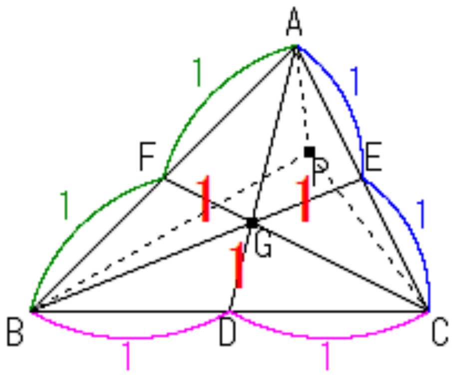
問題 1 三角形 ABC において，$\mathrm{PA}^2+\mathrm{PB}^2+\mathrm{PC}^2$ を最小にする点 P と最小値を求めなさい．

解答 1

$$
\begin{aligned}
&\mathrm{PA}^2+\mathrm{PB}^2+\mathrm{PC}^2\\
=&\left|\overrightarrow{\boldsymbol{p}}-\overrightarrow{\boldsymbol{a}}\right|^2+\left|\overrightarrow{\boldsymbol{p}}-\overrightarrow{\boldsymbol{b}}\right|^2+\left|\overrightarrow{\boldsymbol{p}}-\overrightarrow{\boldsymbol{c}}\right|^2\\
=&\left|\overrightarrow{\boldsymbol{p}}\right|^2-2\,\overrightarrow{\boldsymbol{p}}\cdot\overrightarrow{\boldsymbol{a}}+\left|\overrightarrow{\boldsymbol{a}}\right|^2+\left|\overrightarrow{\boldsymbol{p}}\right|^2-2\,\overrightarrow{\boldsymbol{p}}\cdot\overrightarrow{\boldsymbol{b}}+\left|\overrightarrow{\boldsymbol{b}}\right|^2+\left|\overrightarrow{\boldsymbol{p}}\right|^2-2\,\overrightarrow{\boldsymbol{p}}\cdot\overrightarrow{\boldsymbol{c}}+\left|\overrightarrow{\boldsymbol{c}}\right|^2\\
=&3\left|\overrightarrow{\boldsymbol{p}}\right|^2-2\,\overrightarrow{\boldsymbol{p}}\cdot(\overrightarrow{\boldsymbol{a}}+\overrightarrow{\boldsymbol{b}}+\overrightarrow{\boldsymbol{c}})+\left|\overrightarrow{\boldsymbol{a}}\right|^2+\left|\overrightarrow{\boldsymbol{b}}\right|^2+\left|\overrightarrow{\boldsymbol{c}}\right|^2\\
=&3\left|\overrightarrow{\boldsymbol{p}}-\frac{\overrightarrow{\boldsymbol{a}}+\overrightarrow{\boldsymbol{b}}+\overrightarrow{\boldsymbol{c}}}{3}\right|^2-\frac{\left|\overrightarrow{\boldsymbol{a}}+\overrightarrow{\boldsymbol{b}}+\overrightarrow{\boldsymbol{c}}\right|^2}{3}+\left|\overrightarrow{\boldsymbol{a}}\right|^2+\left|\overrightarrow{\boldsymbol{b}}\right|^2+\left|\overrightarrow{\boldsymbol{c}}\right|^2\\
&\qquad\left(\overrightarrow{\boldsymbol{g}}=\frac{\overrightarrow{\boldsymbol{a}}+\overrightarrow{\boldsymbol{b}}+\overrightarrow{\boldsymbol{c}}}{3}\ \text{とおく}\right)\\
=&3\left|\overrightarrow{\boldsymbol{p}}-\overrightarrow{\boldsymbol{g}}\right|^2+\frac{2\left(\left|\overrightarrow{\boldsymbol{a}}\right|^2+\left|\overrightarrow{\boldsymbol{b}}\right|^2+\left|\overrightarrow{\boldsymbol{c}}\right|^2-\overrightarrow{\boldsymbol{a}}\cdot\overrightarrow{\boldsymbol{b}}-\overrightarrow{\boldsymbol{b}}\cdot\overrightarrow{\boldsymbol{c}}-\overrightarrow{\boldsymbol{c}}\cdot\overrightarrow{\boldsymbol{a}}\right)}{3}\\
=&3\left|\overrightarrow{\boldsymbol{p}}-\overrightarrow{\boldsymbol{g}}\right|^2+\frac{\left|\overrightarrow{\boldsymbol{a}}-\overrightarrow{\boldsymbol{b}}\right|^2+\left|\overrightarrow{\boldsymbol{b}}-\overrightarrow{\boldsymbol{c}}\right|^2+\left|\overrightarrow{\boldsymbol{c}}-\overrightarrow{\boldsymbol{a}}\right|^2}{3}\\
=&3\mathrm{PG}^2+\frac{\mathrm{AB}^2+\mathrm{BC}^2+\mathrm{CA}^2}{3}
\end{aligned}
$$

したがって，P が G（三角形 ABC の重心）のとき，最小値 $\dfrac{\mathrm{AB}^2+\mathrm{BC}^2+\mathrm{CA}^2}{3}$ になる．

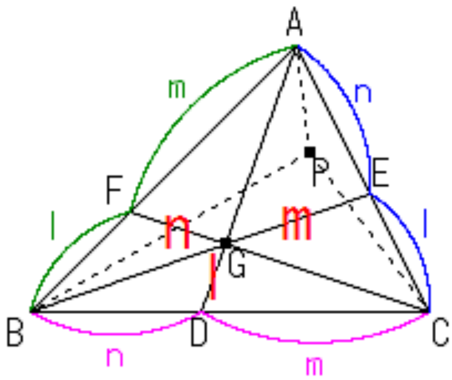


問題 2 三角形 ABC において，$l\mathrm{PA}^2 + m\mathrm{PB}^2 + n\mathrm{PC}^2$ を最小にする点 P と最小値を求めなさい．

解答 2

$$
\begin{aligned}
& l\mathrm{PA}^2 + m\mathrm{PB}^2 + n\mathrm{PC}^2 \\
=& l\left|\overrightarrow{\boldsymbol{p}} - \overrightarrow{\boldsymbol{a}}\right|^2 + m\left|\overrightarrow{\boldsymbol{p}} - \overrightarrow{\boldsymbol{b}}\right|^2 + n\left|\overrightarrow{\boldsymbol{p}} - \overrightarrow{\boldsymbol{c}}\right|^2 \\
=& l\left(\left|\overrightarrow{\boldsymbol{p}}\right|^2 - 2\overrightarrow{\boldsymbol{p}}\cdot\overrightarrow{\boldsymbol{a}} + \left|\overrightarrow{\boldsymbol{a}}\right|^2\right) + m\left(\left|\overrightarrow{\boldsymbol{p}}\right|^2 - 2\overrightarrow{\boldsymbol{p}}\cdot\overrightarrow{\boldsymbol{b}} + \left|\overrightarrow{\boldsymbol{b}}\right|^2\right) + n\left(\left|\overrightarrow{\boldsymbol{p}}\right|^2 - 2\overrightarrow{\boldsymbol{p}}\cdot\overrightarrow{\boldsymbol{c}} + \left|\overrightarrow{\boldsymbol{c}}\right|^2\right) \\
=& (l+m+n)\left|\overrightarrow{\boldsymbol{p}}\right|^2 - 2\overrightarrow{\boldsymbol{p}}\cdot(l\overrightarrow{\boldsymbol{a}} + m\overrightarrow{\boldsymbol{b}} + n\overrightarrow{\boldsymbol{c}}) + l\left|\overrightarrow{\boldsymbol{a}}\right|^2 + m\left|\overrightarrow{\boldsymbol{b}}\right|^2 + n\left|\overrightarrow{\boldsymbol{c}}\right|^2 \\
=& (l+m+n)\left|\overrightarrow{\boldsymbol{p}} - \frac{l\overrightarrow{\boldsymbol{a}} + m\overrightarrow{\boldsymbol{b}} + n\overrightarrow{\boldsymbol{c}}}{l+m+n}\right|^2 - \frac{\left|l\overrightarrow{\boldsymbol{a}} + m\overrightarrow{\boldsymbol{b}} + n\overrightarrow{\boldsymbol{c}}\right|^2}{l+m+n} + l\left|\overrightarrow{\boldsymbol{a}}\right|^2 + m\left|\overrightarrow{\boldsymbol{b}}\right|^2 + n\left|\overrightarrow{\boldsymbol{c}}\right|^2 \\
& \qquad \left(\overrightarrow{\boldsymbol{g}} = \frac{l\overrightarrow{\boldsymbol{a}} + m\overrightarrow{\boldsymbol{b}} + n\overrightarrow{\boldsymbol{c}}}{l+m+n} \text{ とおく}\right) \\
=& (l+m+n)\left|\overrightarrow{\boldsymbol{p}} - \overrightarrow{\boldsymbol{g}}\right|^2 + \frac{lm\left|\overrightarrow{\boldsymbol{a}} - \overrightarrow{\boldsymbol{b}}\right|^2 + mn\left|\overrightarrow{\boldsymbol{b}} - \overrightarrow{\boldsymbol{c}}\right|^2 + nl\left|\overrightarrow{\boldsymbol{c}} - \overrightarrow{\boldsymbol{a}}\right|^2}{l+m+n} \\
=& (l+m+n)\mathrm{PG}^2 + \frac{lm\mathrm{AB}^2 + mn\mathrm{BC}^2 + nl\mathrm{CA}^2}{l+m+n}
\end{aligned}
$$

したがって，P が G のとき，最小値 $\dfrac{lm\mathrm{AB}^2 + mn\mathrm{BC}^2 + nl\mathrm{CA}^2}{l+m+n}$ になる．



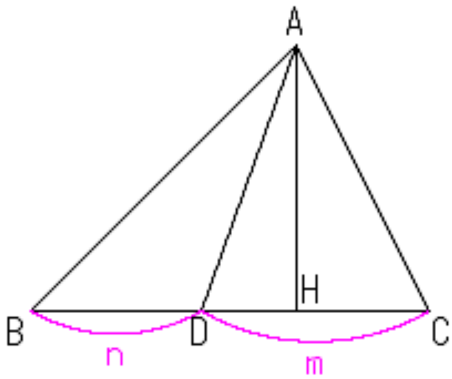
ベクトルを使わない解答

補題（中線定理の拡張）

三角形 ABC において，辺 BC の $n:m$ の内分点を D とすると，

$$m\mathrm{AB}^2+n\mathrm{AC}^2=(m+n)\left(\mathrm{AD}^2+\mathrm{BD}\cdot\mathrm{CD}\right)=(m+n)\mathrm{AD}^2+\frac{mn}{m+n}\mathrm{BC}^2$$

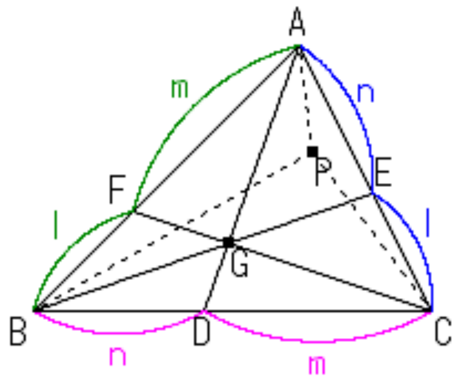
証明



頂点 A から辺 BC に下ろした垂線の足を H とする．

$$
\begin{aligned}
&m\mathrm{AB}^2+n\mathrm{AC}^2\\
=&m\left(\mathrm{AH}^2+\mathrm{BH}^2\right)+n\left(\mathrm{AH}^2+\mathrm{CH}^2\right)\\
=&m\left(\mathrm{AD}^2-\mathrm{DH}^2+(\mathrm{BD}+\mathrm{DH})^2\right)+n\left(\mathrm{AD}^2-\mathrm{DH}^2+(\mathrm{CD}-\mathrm{DH})^2\right)\\
=&(m+n)\mathrm{AD}^2+m\mathrm{BD}^2+n\mathrm{CD}^2+2\mathrm{DH}(m\mathrm{BD}-n\mathrm{CD})\\
&\quad (m\mathrm{BD}=n\mathrm{CD}\ \text{だから})\\
=&(m+n)\mathrm{AD}^2+n\mathrm{CD}\cdot\mathrm{BD}+m\mathrm{BD}\cdot\mathrm{CD}\\
=&(m+n)\left(\mathrm{AD}^2+\mathrm{BD}\cdot\mathrm{CD}\right)\\
=&(m+n)\mathrm{AD}^2+\frac{mn}{m+n}\mathrm{BC}^2
\end{aligned}
$$

解答 3



三角形 PBC において　$m\mathrm{PB}^2+n\mathrm{PC}^2=(m+n)\mathrm{PD}^2+\dfrac{mn}{m+n}\mathrm{BC}^2$

三角形 PCD において　$n\mathrm{PC}^2+l\mathrm{PA}^2=(n+l)\mathrm{PE}^2+\dfrac{nl}{n+l}\mathrm{CA}^2$

三角形 PAB において　$l\mathrm{PA}^2+m\mathrm{PB}^2=(l+m)\mathrm{PF}^2+\dfrac{lm}{l+m}\mathrm{AB}^2$

三角形 PAD において　$l\mathrm{PA}^2+(m+n)\mathrm{PD}^2=(l+m+n)\mathrm{PG}^2+\dfrac{l(m+n)}{l+m+n}\mathrm{AD}^2$

三角形 PBE において　$m\mathrm{PB}^2+(n+l)\mathrm{PE}^2=(m+n+l)\mathrm{PG}^2+\dfrac{m(n+l)}{m+n+l}\mathrm{BE}^2$

三角形 PCF において　$n\mathrm{PC}^2+(l+m)\mathrm{PF}^2=(n+l+m)\mathrm{PG}^2+\dfrac{n(l+m)}{n+l+m}\mathrm{CF}^2$

これらを足し合わせると

$$
\begin{aligned}
3\left(l\mathrm{PA}^2+m\mathrm{PB}^2+n\mathrm{PC}^2\right)=&3(l+m+n)\mathrm{PG}^2+\frac{mn}{m+n}\mathrm{BC}^2+\frac{nl}{n+l}\mathrm{CA}^2+\frac{lm}{l+m}\mathrm{AB}^2\\
&+\frac{l(m+n)}{l+m+n}\mathrm{AD}^2+\frac{m(n+l)}{l+m+n}\mathrm{BE}^2+\frac{n(l+m)}{l+m+n}\mathrm{CF}^2
\end{aligned}
$$

ゆえに，$l\mathrm{PA}^2+m\mathrm{PB}^2+n\mathrm{PC}^2$ は $\mathrm{P}=\mathrm{G}$ のとき最小になり，その値は

$$
\begin{aligned}
&\frac{1}{3}\left(\frac{mn}{m+n}\mathrm{BC}^2+\frac{nl}{n+l}\mathrm{CA}^2+\frac{lm}{l+m}\mathrm{AB}^2\right.\\
&\qquad\left.+\frac{l(m+n)}{l+m+n}\mathrm{AD}^2+\frac{m(n+l)}{l+m+n}\mathrm{BE}^2+\frac{n(l+m)}{l+m+n}\mathrm{CF}^2\right)\\
=&\frac{1}{3}\left(\frac{mn}{m+n}\mathrm{BC}^2+\frac{nl}{n+l}\mathrm{CA}^2+\frac{lm}{l+m}\mathrm{AB}^2\right.\\
&\qquad+\frac{l}{l+m+n}\left(m\mathrm{AB}^2+n\mathrm{AC}^2-\frac{mn}{m+n}\mathrm{BC}^2\right)\\
&\qquad+\frac{m}{l+m+n}\left(n\mathrm{BC}^2+l\mathrm{BA}^2-\frac{nl}{n+l}\mathrm{CA}^2\right)\\
&\qquad\left.+\frac{n}{l+m+n}\left(l\mathrm{CA}^2+m\mathrm{CB}^2-\frac{lm}{l+m}\mathrm{AB}^2\right)\right)\\
=&\frac{mn\mathrm{BC}^2+nl\mathrm{CA}^2+lm\mathrm{AB}^2}{l+m+n}
\end{aligned}
$$