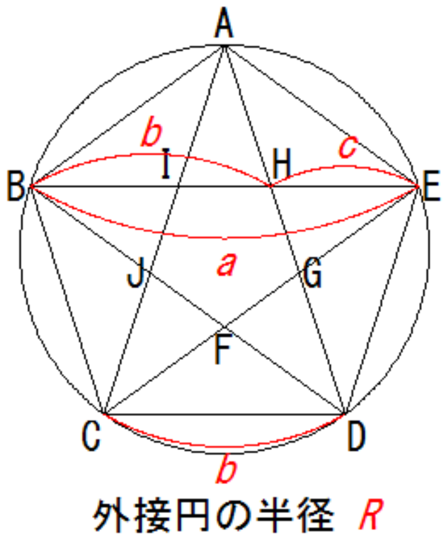
1　正五角形・星芒形



1.1　a, b, c の関係

$$\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CD} = \overset{\frown}{DE} = \overset{\frown}{EA}$$

ゆえに，三角形 ACD において

$$\angle\mathrm{ADB} = \angle\mathrm{BDC} = \angle\mathrm{CAD} = \angle\mathrm{DCE} = \angle\mathrm{ECA} = \frac{180^\circ}{5} = 36^\circ$$

頂角が同じ二等辺三角形は相似である

$$\triangle\mathrm{ABE} \propto \triangle\mathrm{FCD}$$
$$\mathrm{BE} : \mathrm{CD} = \mathrm{AB} : \mathrm{FC}$$
$$\begin{cases} a : b = b : c \\ a = b + c \end{cases} \quad \cdots \quad \text{この比を黄金比といい，}\phi\text{ で表す.}$$
$$a = b\phi\,,\quad b = c\phi\,,\quad a = c\phi^2$$

1.2　黄金比 ϕ の値

$a = b + c$ より

$$\phi^2 - \phi - 1 = 0 \qquad \phi^{-2} + \phi^{-1} - 1 = 0$$
$$\phi = \frac{1+\sqrt{5}}{2} = \frac{\sqrt{5}+1}{2} \qquad \phi^{-1} = \frac{-1+\sqrt{5}}{2} = \frac{\sqrt{5}-1}{2}$$

1.3 ϕ^n を ϕ の 1 次式で表す

ϕ^n を ϕ の 1 次式で表すと，値を計算するとき楽である．

$$
\begin{aligned}
\phi^2 &= \phi+1 &&= \phi+1 \\
\phi^3 &= \phi^2+\phi &&= 2\phi+1 \\
\phi^4 &= \phi^3+\phi^2 &&= 3\phi+2 \\
\phi^5 &= \phi^4+\phi^3 &&= 5\phi+3 \\
&\vdots
\end{aligned}
\qquad
\begin{aligned}
\phi^{-1} &= \phi-1 &&= \phi-1 \\
\phi^{-2} &= 1-\phi^{-1} &&= -\phi+2 \\
\phi^{-3} &= \phi^{-1}-\phi^{-2} &&= 2\phi-3 \\
\phi^{-4} &= \phi^{-2}-\phi^{-3} &&= -3\phi+5 \\
&\vdots
\end{aligned}
$$

次の漸化式で定まる数列を**フィボナッチ数列**という．

$$
\begin{cases}
f_0 = 0\,, \quad f_1 = 1 \\
f_n = f_{n-1} + f_{n-2}\ (n = 2, 3, 4, \dots) \\
f_n = f_{n+2} - f_{n+1}\ (n = -1, -2, -3, \dots)
\end{cases}
$$

n	$\cdots$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	$\cdots$
f_n	$\cdots$	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	$\cdots$

$$f_{-n} = (-1)^{n-1} f_n$$

ϕ^n はフィボナッチ数を用いて次のように表される．

$$\phi^n = f_n\,\phi + f_{n-1}$$

逆に，フィボナッチ数を ϕ を用いて表せる．

$$
\begin{aligned}
&\phi^{-n} = f_{-n}\,\phi + f_{-n-1} = (-1)^{n-1} f_n \phi + (-1)^{n-2} f_{n+1} \\
&(-\phi^{-1})^n = (-1)^n\,\phi^{-n} = -f_n\,\phi + f_{n+1} \\
&\phi^n - (-\phi^{-1})^n = 2 f_n\,\phi + f_{n-1} - f_{n+1} = 2\phi\,f_n + f_{n-1} - (f_{n-1} + f_n) = (2\phi - 1) f_n = \sqrt{5}\,f_n \\
&\therefore\ f_n = \frac{\phi^n - (-\phi^{-1})^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)
\end{aligned}
$$

1.4 よく使う変形

$$
\begin{aligned}
&2 + \phi = \frac{5+\sqrt{5}}{2} = \sqrt{5}\,\phi \\
&3 - \phi = \frac{5-\sqrt{5}}{2} = \sqrt{5}\,\phi^{-1}
\end{aligned}
$$

1.5　$36^\circ, 72^\circ$ の三角比

$\triangle$ABE において

$$\mathrm{BE} = 2\mathrm{AB}\cos 36^\circ$$

$$\cos 36^\circ = \frac{a}{2b} = \frac{\phi}{2} = \frac{\sqrt{5}+1}{4}$$

$$\sin 36^\circ = \sqrt{\frac{4-\phi^2}{4}} = \sqrt{\frac{3-\phi}{4}} = \sqrt{\frac{\sqrt{5}\,\phi^{-1}}{4}} = \sqrt{\frac{5-\sqrt{5}}{8}}$$

$\triangle$ACD において

$$\mathrm{CD} = 2\mathrm{AC}\cos 72^\circ$$

$$\cos 72^\circ = \frac{b}{2a} = \frac{\phi^{-1}}{2} = \frac{\sqrt{5}-1}{4}$$

$$\sin 72^\circ = \sqrt{\frac{4-\phi^{-2}}{4}} = \sqrt{\frac{2+\phi}{4}} = \sqrt{\frac{\sqrt{5}\,\phi}{4}} = \sqrt{\frac{5+\sqrt{5}}{8}}$$

1.6　a, b, c と R の相互関係

正弦定理より

$$a = 2R\sin 72^\circ = \sqrt{\sqrt{5}\,\phi}\ R = \sqrt{\frac{5+\sqrt{5}}{2}}\ R \qquad R = \sqrt{\frac{\sqrt{5}\,\phi^{-1}}{5}}\ a = \sqrt{\frac{5-\sqrt{5}}{10}}\ a$$

$$b = 2R\sin 36^\circ = \sqrt{\sqrt{5}\,\phi^{-1}}\ R = \sqrt{\frac{5-\sqrt{5}}{2}}\ R \qquad R = \sqrt{\frac{\sqrt{5}\,\phi}{5}}\ b = \sqrt{\frac{5+\sqrt{5}}{10}}\ b$$

$$c = \phi^{-1}\,b = \sqrt{\sqrt{5}\,\phi^{-3}}\ R = \sqrt{5-2\sqrt{5}}\ R \qquad R = \sqrt{\frac{\sqrt{5}\,\phi^{3}}{5}}\ c = \sqrt{\frac{5+2\sqrt{5}}{5}}\ c$$

1.7 三角形 ACD, ABE の面積

面積の公式より

$$\triangle\mathrm{ACD} = \frac{1}{2}a^2\sin 36^\circ = \frac{\sqrt{\sqrt{5}\,\phi^{-1}}}{4}\,a^2 = \sqrt{\frac{5-\sqrt{5}}{32}}\,a^2$$

$$= \frac{\sqrt{\sqrt{5}\,\phi^{-1}}\;\phi^2}{4}\,b^2 = \frac{\sqrt{\sqrt{5}\,\phi^3}}{4}\,b^2 = \sqrt{\frac{5+2\sqrt{5}}{16}}\,b^2$$

$$= \frac{\sqrt{\sqrt{5}\,\phi^{-1}}\;\phi^4}{4}\,c^2 = \frac{\sqrt{\sqrt{5}\,\phi^7}}{4}\,c^2 = \sqrt{\frac{65+29\sqrt{5}}{32}}\,c^2$$

$$= \frac{\sqrt{\sqrt{5}\,\phi^{-1}}\;\sqrt{5}\,\phi}{4}\,R^2 = \frac{\sqrt{5\sqrt{5}\,\phi}}{4}\,R^2 = \sqrt{\frac{5(5+\sqrt{5}\,)}{32}}\,R^2$$

$$\triangle\mathrm{ABE} = \frac{1}{2}ab\sin 36^\circ = \frac{\sqrt{\sqrt{5}\,\phi^{-1}}\;\phi^{-1}}{4}\,a^2 = \frac{\sqrt{\sqrt{5}\,\phi^{-3}}}{4}\,a^2 = \sqrt{\frac{5-2\sqrt{5}}{16}}\,a^2$$

$$= \frac{\sqrt{\sqrt{5}\,\phi^{-1}}\;\phi}{4}\,b^2 = \frac{\sqrt{\sqrt{5}\,\phi}}{4}\,b^2 = \sqrt{\frac{5+\sqrt{5}}{32}}\,b^2$$

$$= \frac{\sqrt{\sqrt{5}\,\phi^{-1}}\;\phi^3}{4}\,c^2 = \frac{\sqrt{\sqrt{5}\,\phi^5}}{4}\,c^2 = \sqrt{\frac{25+11\sqrt{5}}{32}}\,c^2$$

$$= \frac{\sqrt{\sqrt{5}\,\phi^{-1}}\,\sqrt{5}}{4}\,R^2 = \frac{\sqrt{5\sqrt{5}\,\phi^{-1}}}{4}\,R^2 = \sqrt{\frac{5(5-\sqrt{5}\,)}{32}}\,R^2$$

1.8 五角形 ABCDE の面積 S_1

$$S_1 = \triangle\mathrm{ACD} + 2\triangle\mathrm{ABE} = (1+2\phi^{-1})\triangle\mathrm{ACD} = \sqrt{5}\,\triangle\mathrm{ACD}$$

$$= \frac{\sqrt{5\sqrt{5}\,\phi^{-1}}}{4}\,a^2 = \sqrt{\frac{5(5-\sqrt{5}\,)}{32}}\,a^2$$

$$= \frac{\sqrt{5\sqrt{5}\,\phi^3}}{4}\,b^2 = \sqrt{\frac{5(5+2\sqrt{5}\,)}{16}}\,b^2$$

$$= \frac{\sqrt{5\sqrt{5}\,\phi^7}}{4}\,c^2 = \sqrt{\frac{5(65+29\sqrt{5}\,)}{32}}\,c^2$$

$$= \frac{\sqrt{25\sqrt{5}\,\phi}}{4}\,R^2 = \sqrt{\frac{25(5+\sqrt{5}\,)}{32}}\,R^2$$

異なる求め方（外接円の中心を O とする）

$$S_1 = 5\triangle\mathrm{OCD} = 5\cdot\frac{1}{2}R^2\sin 72^\circ = \frac{5\sqrt{\sqrt{5}\,\phi}}{4}\,R^2 = \sqrt{\frac{25(5+\sqrt{5}\,)}{32}}\,R^2$$

1.9 三角形 FCD の面積

$\triangle\mathrm{FCD} \propto \triangle\mathrm{ABE}$ (相似比 $b : a$)

$$\begin{aligned}
\triangle\mathrm{FCD} = \phi^{-2}\triangle\mathrm{ABE} &= \frac{\sqrt{\sqrt{5}\,\phi^{-7}}}{4}\,a^2 &&= \sqrt{\frac{65-29\sqrt{5}}{32}}\,a^2 \\
&= \frac{\sqrt{\sqrt{5}\,\phi^{-3}}}{4}\,b^2 &&= \sqrt{\frac{5-2\sqrt{5}}{16}}\,b^2 \\
&= \frac{\sqrt{\sqrt{5}\,\phi}}{4}\,c^2 &&= \sqrt{\frac{5+\sqrt{5}}{32}}\,c^2 \\
&= \frac{\sqrt{5\sqrt{5}\,\phi^{-5}}}{4}\,R^2 &&= \sqrt{\frac{5(25-11\sqrt{5}\,)}{32}}\,R^2
\end{aligned}$$

1.10 星芒形 AIBJCFDGEH の面積 S_2

$$\begin{aligned}
S_2 &= 2\triangle\mathrm{ABE} + \triangle\mathrm{ACD} - 5\triangle\mathrm{FCD} \\
&= (2\phi^2 + \phi^3 - 5)\triangle\mathrm{FCD} \\
&= (4\phi - 2)\triangle\mathrm{FCD} \\
&= 2\sqrt{5}\,\triangle\mathrm{FCD}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{5\sqrt{5}\,\phi^{-7}}}{2}\,a^2 &&= \sqrt{\frac{5(65-29\sqrt{5}\,)}{8}}\,a^2 \\
&= \frac{\sqrt{5\sqrt{5}\,\phi^{-3}}}{2}\,b^2 &&= \sqrt{\frac{5(5-2\sqrt{5}\,)}{4}}\,b^2 \\
&= \frac{\sqrt{5\sqrt{5}\,\phi}}{2}\,c^2 &&= \sqrt{\frac{5(5+\sqrt{5}\,)}{8}}\,c^2 \\
&= \frac{\sqrt{25\sqrt{5}\,\phi^{-5}}}{2}\,R^2 &&= \sqrt{\frac{25(25-11\sqrt{5}\,)}{8}}\,R^2
\end{aligned}$$