# 1　不定積分 $I = \displaystyle\int \frac{1}{\sqrt{2-x-x^2}}\,dx$

## 1.1　解答 1

$x + \frac{1}{2} = \frac{3}{2}\sin\theta \quad \left(-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\right)$ と置換する。

$$
\begin{aligned}
dx &= \frac{3}{2}\cos\theta\,d\theta \\
\sqrt{2-x-x^2} &= \sqrt{\frac{9}{4} - \left(x+\frac{1}{2}\right)^2} = \sqrt{\frac{9}{4}(1-\sin^2\theta)} = \frac{3}{2}\cos\theta \\
I &= \int \frac{1}{\frac{3}{2}\cos\theta}\,\frac{3}{2}\cos\theta d\theta = \int 1\,d\theta = \theta + C = \sin^{-1}\left(\frac{2x+1}{3}\right) + C
\end{aligned}
$$

## 1.2　解答 2

$t = \sqrt{\frac{2+x}{1-x}}$ と置換する。

$$
\begin{aligned}
1+t^2 &= \frac{3}{1-x} \qquad 1-x = \frac{3}{1+t^2} \\
-dx &= -\frac{6t}{(1+t^2)^2}dt \\
\sqrt{2-x-x^2} &= \sqrt{(1-x)(2+x)} = (1-x)\sqrt{\frac{2+x}{1-x}} = \frac{3}{1+t^2}\,t \\
I &= \int \frac{1+t^2}{3t}\,\frac{6t}{(1+t^2)^2}dt = \int \frac{2}{1+t^2}\,dt
\end{aligned}
$$

更に，$t = \tan\phi \quad \left(-\frac{\pi}{2} < \phi < \frac{\pi}{2}\right)$ と置換する。

$$
\begin{aligned}
dt &= (1+\tan^2\phi)d\phi \\
I &= \int \frac{2}{1+\tan^2\phi}\,(1+\tan^2\phi)d\phi = \int 2\,d\phi = 2\phi + C = 2\tan^{-1}\left(\sqrt{\frac{2+x}{1-x}}\right) + C
\end{aligned}
$$

## 1.3　$\theta$ と $\phi$ の関係

$\sin\theta = \frac{2x+1}{3} \quad \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ より

$$
\begin{aligned}
\tan\theta &= \frac{2x+1}{\sqrt{3^2-(2x+1)^2}} = \frac{(2+x)-(1-x)}{2\sqrt{(2+x)(1-x)}} = \frac{1}{2}\left(\sqrt{\frac{2+x}{1-x}} - \sqrt{\frac{1-x}{2+x}}\right) \\
&= \frac{1}{2}\left(\tan\phi - \frac{1}{\tan\phi}\right) = -\frac{1-\tan^2\phi}{2\tan\phi} = -\frac{1}{\tan 2\phi} = \tan\left(2\phi - \frac{\pi}{2}\right) \\
\therefore\ \theta &= 2\phi - \frac{\pi}{2}
\end{aligned}
$$