Σ の公式

公式 1

$$\sum_{k=1}^{n} k = \frac{1}{2}(n+1)n$$

$$\sum_{k=1}^{n} k(k-1) = \frac{1}{3}(n+1)n(n-1)$$

$$\sum_{k=1}^{n} k(k-1)(k-2) = \frac{1}{4}(n+1)n(n-1)(n-2)$$

$$\vdots$$

$$\sum_{k=1}^{n} k(k-1)(k-2)\cdots(k-p) = \frac{1}{p+2}(n+1)n(n-1)(n-2)\cdots(n-p)$$

$p=2$ のとき

$$\begin{aligned}
&\sum_{k=1}^{n} k(k-1)(k-2) \\
&= \sum_{k=1}^{n} k(k-1)(k-2)\big((k+1)-(k-3)\big)\cdot\frac{1}{4} \\
&= \frac{1}{4}\sum_{k=1}^{n}\big((k+1)k(k-1)(k-2)-k(k-1)(k-2)(k-3)\big) \\
&= \frac{1}{4}\big((n+1)n(n-1)(n-2)-1\cdot 0\cdot(-1)\cdot(-2)\big) \\
&= \frac{1}{4}(n+1)n(n-1)(n-2)
\end{aligned}$$

一般に

$$\begin{aligned}
&\sum_{k=1}^{n} k(k-1)(k-2)\cdots(k-p) \\
&= \sum_{k=1}^{n} k(k-1)(k-2)\cdots(k-p)\big((k+1)-(k-p-1)\big)\cdot\frac{1}{p+2} \\
&= \frac{1}{p+2}\sum_{k=1}^{n}\big((k+1)k(k-1)(k-2)\cdots(k-p)-k(k-1)(k-2)\cdots(k-p)(k-p-1)\big) \\
&= \frac{1}{p+2}\big((n+1)n(n-1)(n-2)\cdots(n-p)-1\cdot 0\cdot(-1)\cdot(-2)\cdot\cdots\cdot(-p+1)(-p)\big) \\
&= \frac{1}{p+2}(n+1)n(n-1)(n-2)\cdots(n-p)
\end{aligned}$$

公式 2

$$\sum_{k=1}^{n} 1 = n$$

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$$

$$\sum_{k=1}^{n} k^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

$$\sum_{k=1}^{n} k^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

2 乗の和 $k^2 = k(k-1)+k$ だから

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} \{(k-1)+k\} \\ &= \frac{1}{3}(n+1)n(n-1) + \frac{1}{2}(n+1)n \\ &= \frac{1}{6}n(n+1)\{2(n-1)+3\} \\ &= \frac{1}{6}n(n+1)(2n+1)\end{aligned}$$

3 乗の和 $k^3 = k(k-1)(k-2) + Bk(k-1) + Ak$ とおくと

$k=1 \rightarrow 1{=}A \quad \therefore\ A=1$

$k=2 \rightarrow 8{=}2B+2A \quad \therefore\ B=3$

ゆえに

$$\begin{aligned}\sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} \{k(k-1)(k-2)+3k(k-1)+k\} \\ &= \frac{1}{4}(n+1)n(n-1)(n-2) + \frac{3}{3}(n+1)n(n-1) + \frac{1}{2}(n+1)n \\ &= \frac{1}{4}n(n+1)\{(n-1)(n-2)+4(n-1)+2\} \\ &= \frac{1}{4}n(n+1)(n^2+n) \\ &= \frac{1}{4}n^2(n+1)^2\end{aligned}$$

4 乗の和 $k^4 = k(k-1)(k-2)(k-3) + Ck(k-1)(k-2) + Bk(k-1) + Ak$ とおくと

$$\begin{array}{llll} k=1 & \to & 1=A & \therefore\ A=1 \\ k=2 & \to & 16=2B+2A & \therefore\ B=7 \\ k=3 & \to & 81=6C+6B+3A & \therefore\ C=6 \end{array}$$

ゆえに

$$\begin{aligned}\sum_{k=1}^{n} k^4 &= \sum_{k=1}^{n} \{k(k-1)(k-2)(k-3) + 6k(k-1)(k-2) + 7k(k-1) + k\} \\ &= \quad \frac{1}{5}(n+1)n(n-1)(n-2)(n-3) \\ &\quad + \frac{6}{4}(n+1)n(n-1)(n-2) \\ &\quad + \frac{7}{3}(n+1)n(n-1) \\ &\quad + \frac{1}{2}(n+1)n \\ &= \frac{1}{30}n(n+1)\left\{6(n-1)(n-2)(n-3) + 45(n-1)(n-2) + 70(n-1) + 15\right\} \\ &= \frac{1}{30}n(n+1)(6n^3+9n^2+n-1) \\ &= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)\end{aligned}$$

5 乗の和 $k^5 = k(k-1)(k-2)(k-3)(k-4) + Dk(k-1)(k-2)(k-3) + Ck(k-1)(k-2) + Bk(k-1) + Ak$ とおくと

$$\begin{array}{llll} k=1 & \to & 1=A & \therefore\ A=1 \\ k=2 & \to & 32=2B+2A & \therefore\ B=15 \\ k=3 & \to & 243=6C+6B+3A & \therefore\ C=25 \\ k=4 & \to & 1024=24D+24C+12B+4A & \therefore\ D=10 \end{array}$$

ゆえに

$$\begin{aligned}\sum_{k=1}^{n} k^5 &= \quad \frac{1}{6}(n+1)n(n-1)(n-2)(n-3)(n-4) \\ &\quad + \frac{10}{5}(n+1)n(n-1)(n-2)(n-3) \\ &\quad + \frac{25}{4}(n+1)n(n-1)(n-2) \\ &\quad + \frac{15}{3}(n+1)n(n-1) \\ &\quad + \frac{1}{2}(n+1)n \\ &= \frac{1}{12}n(n+1)\left\{2(n-1)(n-2)(n-3)(n-4) + 24(n-1)(n-2)(n-3)\right. \\ &\qquad\qquad \left. +75(n-1)(n-2) + 60(n-1) + 6\right\} \\ &= \frac{1}{12}n^2(n+1)(2n^3+4n^2+n-1) \\ &= \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)\end{aligned}$$

公式 3

$$\sum_{k=1}^{n} k(n-k) = \frac{1}{3\cdot 2}(n+1)n(n-1)$$

$$\sum_{k=1}^{n} k(k-1)(n-k) = \frac{1}{4\cdot 3}(n+1)n(n-1)(n-2)$$

$$\sum_{k=1}^{n} k(k-1)(k-2)(n-k) = \frac{1}{5\cdot 4}(n+1)n(n-1)(n-2)(n-3)$$

$$\vdots$$

$$\sum_{k=1}^{n} k(k-1)\cdots(k-p+1)(n-k) = \frac{1}{(p+2)(p+1)}(n+1)n(n-1)(n-2)\cdots(n-p)$$

導き方

$$\begin{aligned}
&\sum_{k=1}^{n} k(k-1)(k-2)(n-k) \\
&= \sum_{k=1}^{n} k(k-1)(k-2)\big((n-3)-(k-3)\big) \\
&= (n-3)\sum_{k=1}^{n} k(k-1)(k-2) - \sum_{k=1}^{n} k(k-1)(k-2)(k-3) \\
&= (n-3)\cdot\frac{1}{4}(n+1)n(n-1)(n-2) - \frac{1}{5}(n+1)n(n-1)(n-2)(n-3) \\
&= \frac{1}{5\cdot 4}(n+1)n(n-1)(n-2)(n-3)
\end{aligned}$$

公式 4

$$\sum_{k=1}^{n} k(n-k)(n-k-1) = \frac{2}{4\cdot 3\cdot 2}(n+1)n(n-1)(n-2)$$
$$\sum_{k=1}^{n} k(k-1)(n-k)(n-k-1) = \frac{2}{5\cdot 4\cdot 3}(n+1)n(n-1)(n-2)(n-3)$$
$$\sum_{k=1}^{n} k(k-1)(k-2)(n-k)(n-k-1) = \frac{2}{6\cdot 5\cdot 4}(n+1)n(n-1)(n-2)(n-3)(n-4)$$
$$\vdots$$
$$\sum_{k=1}^{n} k(k-1)\cdots(k-p+2)(n-k)(n-k-1) = \frac{2}{(p+2)(p+1)p}(n+1)n(n-1)(n-2)\cdots(n-p)$$

導き方

$$\begin{aligned}
&\sum_{k=1}^{n} k(k-1)(k-2)(n-k)(n-k-1) \\
&= \sum_{k=1}^{n} k(k-1)(k-2)(n-k)\big((n-4)-(k-3)\big) \\
&= (n-4)\sum_{k=1}^{n} k(k-1)(k-2)(n-k) - \sum_{k=1}^{n} k(k-1)(k-2)(k-3)(n-k) \\
&= (n-4)\cdot\frac{1}{5\cdot 4}(n+1)n(n-1)(n-2)(n-3) - \frac{1}{6\cdot 5}(n+1)n(n-1)(n-2)(n-3)(n-4) \\
&= \frac{2}{6\cdot 5\cdot 4}(n+1)n(n-1)(n-2)(n-3)
\end{aligned}$$

たとえば，公式 4 の 2 番目の公式

$$\sum_{k=1}^{n} k(k-1)(n-k)(n-k-1) = \frac{2}{5\cdot 4\cdot 3}(n+1)n(n-1)(n-2)(n-3)$$

は，両辺を 4 で割ると

$$\sum_{k=1}^{n} \frac{k(k-1)}{2}\frac{(n-k)(n-k-1)}{2} = \frac{(n+1)n(n-1)(n-2)(n-3)}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

すなわち

$$\sum_{k=1}^{n} {}_k\mathrm{C}_2 \cdot {}_{n-k}\mathrm{C}_2 = {}_{n+1}\mathrm{C}_5$$

となる。

公式 1,3,4 は，次の一般形の公式から導かれる。

公式 5

$$\sum_{k=0}^{n} {}_k\mathrm{C}_p \cdot {}_{n-k}\mathrm{C}_q = {}_{n+1}\mathrm{C}_{p+q+1}$$

注 1 $r<0$ のときと $r>n$ のとき，${}_n\mathrm{C}_r = 0$ と定める。

組合せ論による導き方

$\{0,1,2,\ldots,n\}$ から $p+q+1$ 個の数を取り出す組合せの数を考える。

(1) ${}_{n+1}\mathrm{C}_{p+q+1}$ 通り（右辺）。

(2) 取り出した数のうち小さい方から $p+1$ 番目の数が k の場合。

$\{0,1,2,\ldots,k-1\}$ から p 個と，$\{k+1,k+2,k+3,\ldots,n\}$ から q 個取り出しているから，${}_k\mathrm{C}_p \cdot {}_{n-k}\mathrm{C}_q$ 通り。

全部で，$\displaystyle\sum_{k=0}^{n} {}_k\mathrm{C}_p \cdot {}_{n-k}\mathrm{C}_q$ 通り（左辺）。